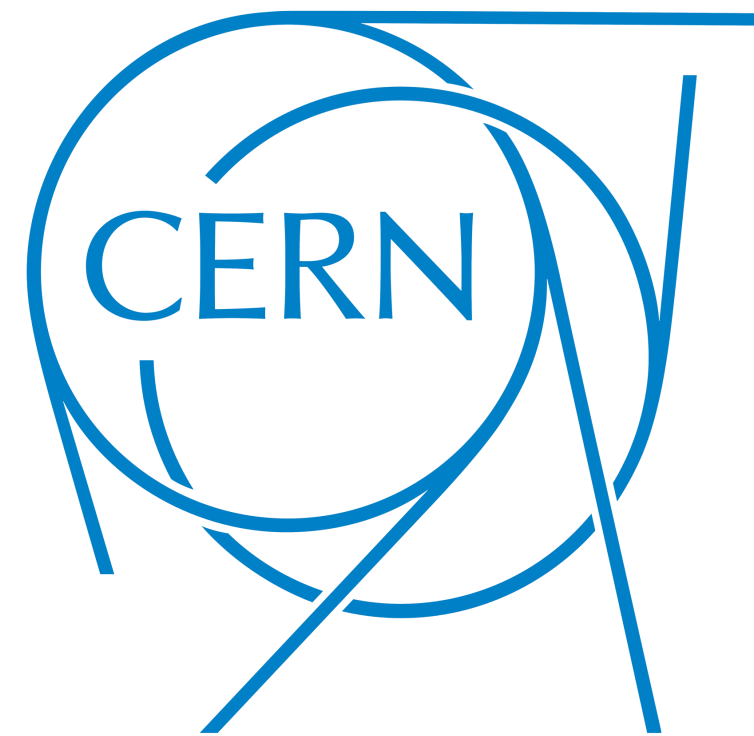




INTO THE (INFRA)RED: SUBTRACTION SCHEMES FOR PRECISION QCD AT HADRON COLLIDERS

CHIARA SIGNORILE-SIGNORILE

Color meets Flavor Symposium,
Siegen, 15.07.2026

MOTIVATIONAL MESSAGE

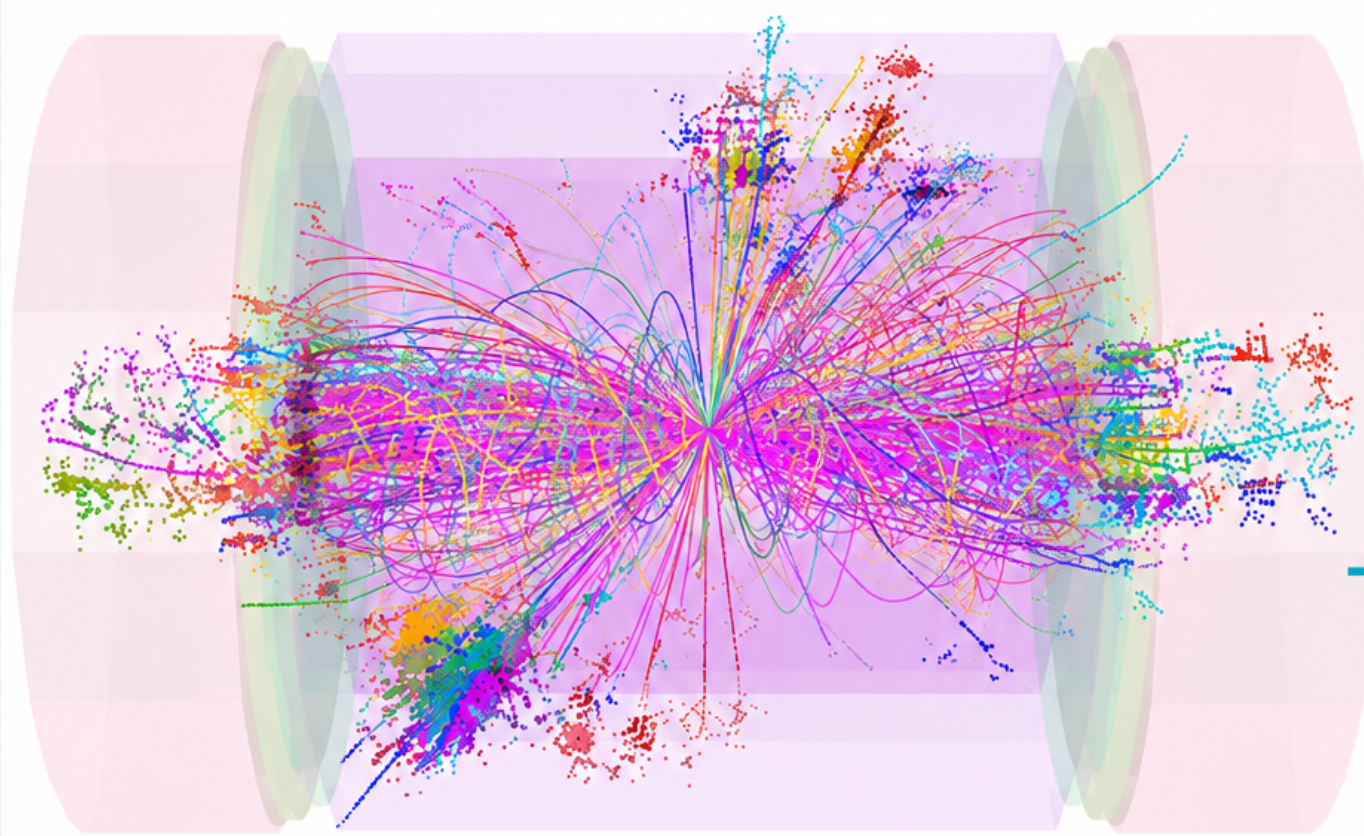
THE STANDARD MODEL LAGRANGIAN

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \\ & + i \bar{\psi} \not{D} \psi \\ & + Y_i \bar{\psi}_i Y_{ij} \psi_j \phi + \text{h.c.} \\ & + |D_\mu \phi|^2 - V(\phi)\end{aligned}$$

● Gauge fields ● Fermions ● Yukawa interactions ● Higgs



A COLLISION AT THE LHC

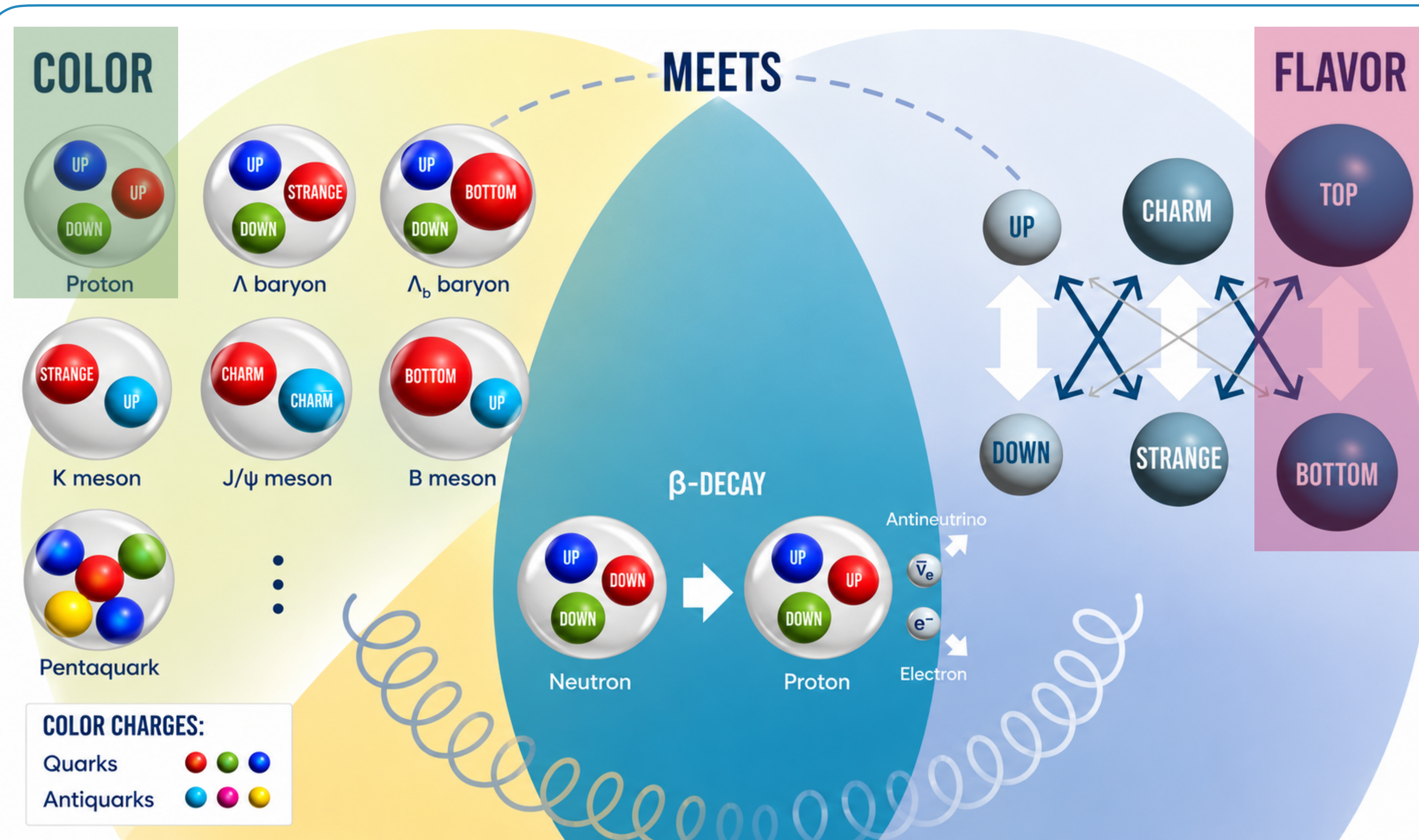


The Standard Model

- ❖ Standard Model of particle physics gives us the “code of the Universe” through a **tiny formula**.
- ❖ It explains outcomes of most current terrestrial experiments and many aspects of the evolution of the Universe (add a grain of gravity!)
- ❖ However, **many questions remain unanswered**: origin of Higgs, EWSB, hierarchy puzzle, neutrino masses, matter/antimatter asymmetry...
- ❖ **Test the Standard Model as thoroughly as possible** stands as one of the main goal of the particle physics community

*“We are not only searching for new physics;
we are searching for something new in physics”*

MOTIVATIONAL MESSAGE



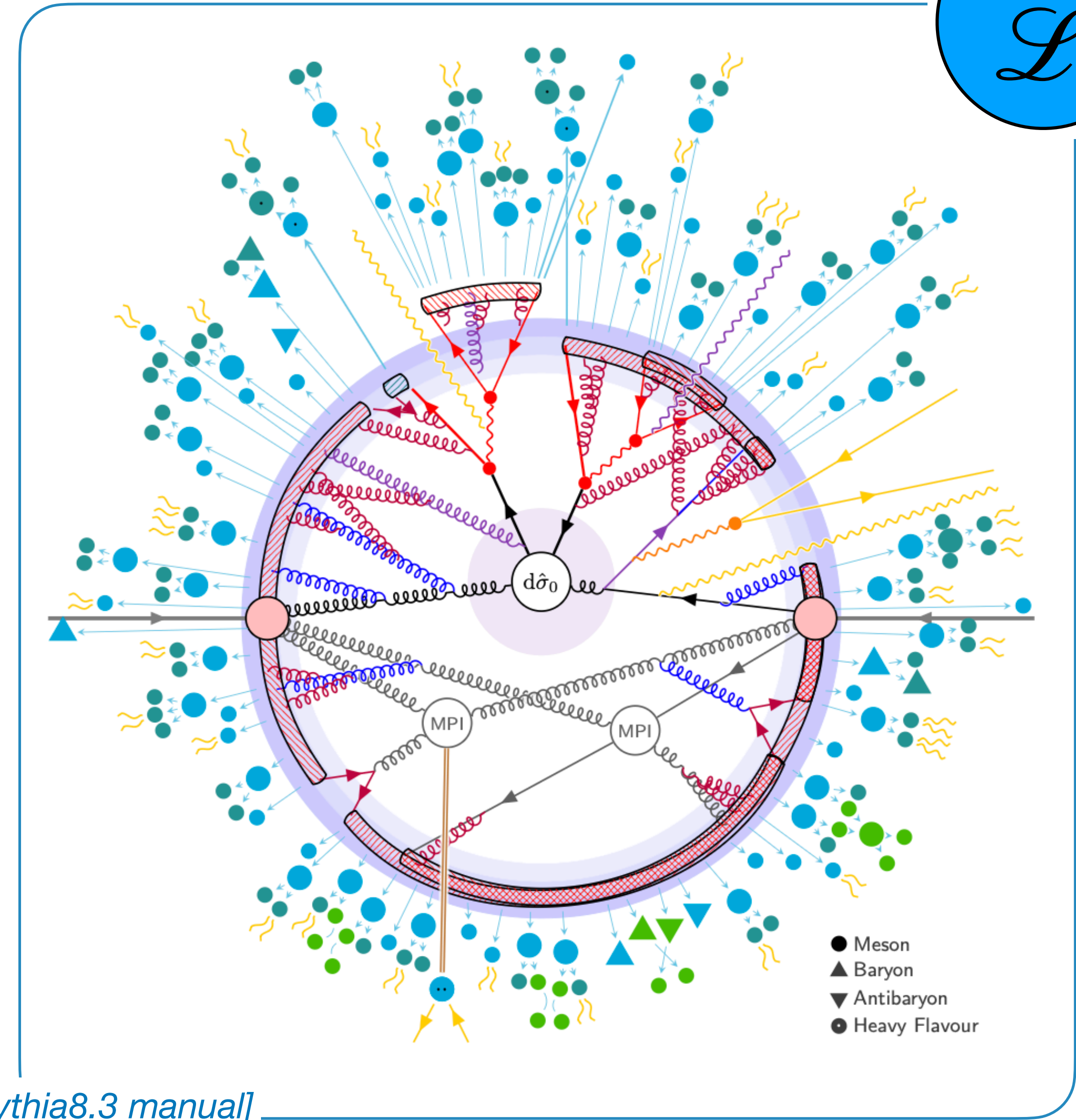
Outlook:

- Develop methods to obtain **high precision theoretical predictions**
- Exploit these methods for a wide range of **phenomenological studies** to improve our **knowledge of particle physics**

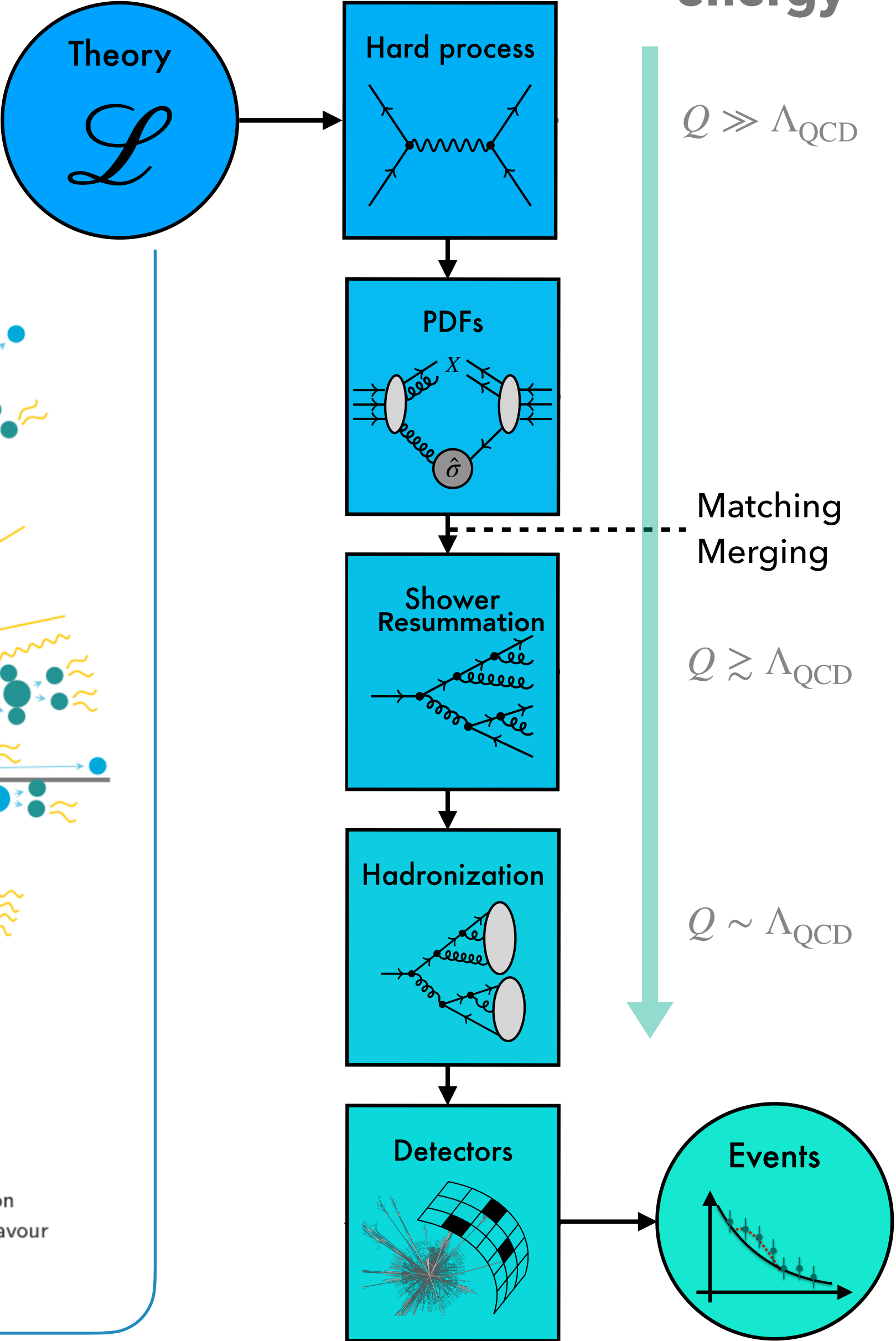
Precision methods in quantum field theory provide a common framework—from infrared singularities in QCD to phenomenological applications in top-quark, electroweak and Higgs physics

THEORETICAL TOOLS

PRECISION&REALISM



[Pythia8.3 manual]



energy

current efforts

F.O. perturbation theory: N^xLO

- ▶ Automate (N)NLO QCD
- ▶ EW and mixed QCD-EW

Matching: $N^xLO + PS_{NyLL}$

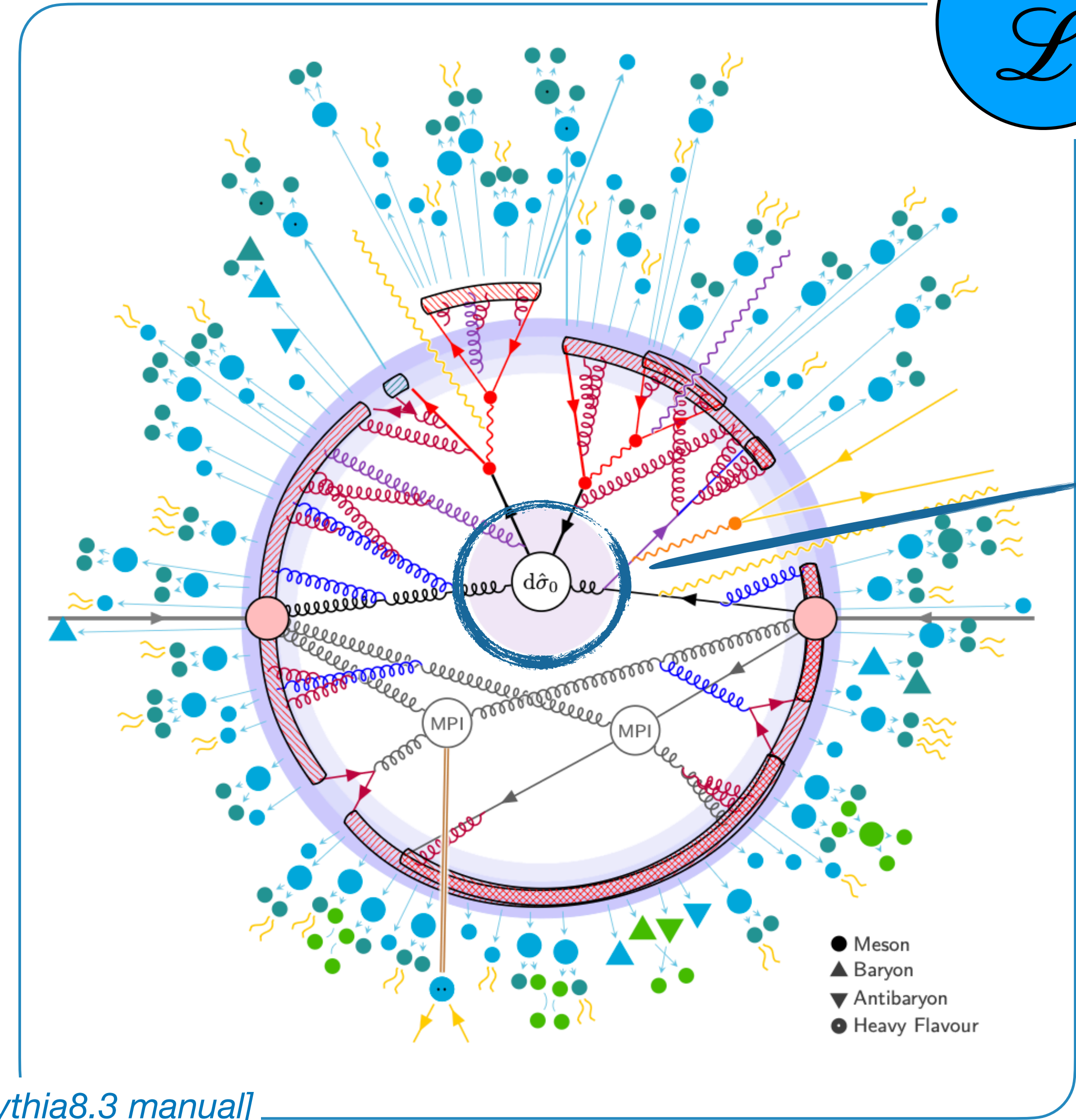
Parton-shower/Resummation: PS_{NyLL}

- ▶ Increasing PS accuracy
- ▶ (N)NLL accurate showers
- ▶ Sub-leading color effects

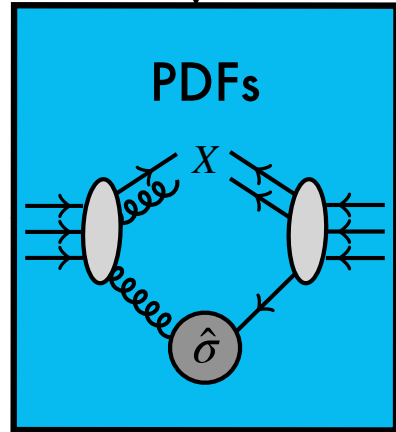
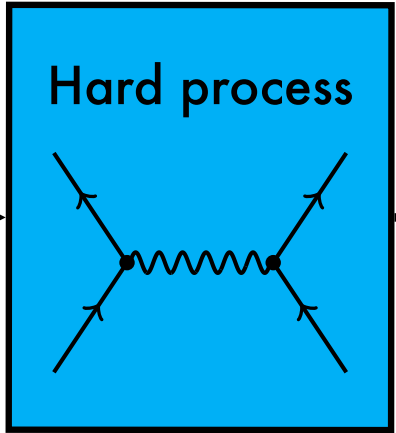
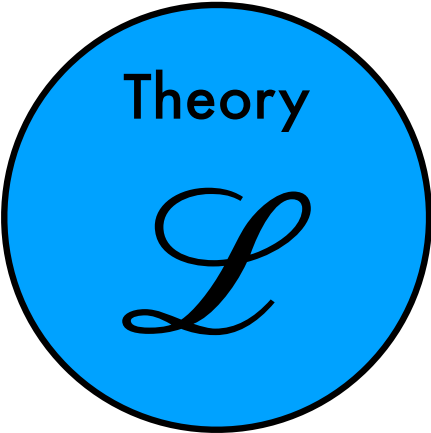
Hadronisation:

- ▶ Refinement of models
- ▶ Improving constraints

PRECISION&REALISM



[Pythia8.3 manual]



energy $Q \gg \Lambda_{\text{QCD}}$ **current efforts** F.O. perturbation theory: N^xLO

Collinear factorisation formula [Collins, Soper, Sterman '04]

$$\frac{d\sigma}{dX} = \sum_{ab} \int dx_1 dx_2 f_{a/A}(x_1, \mu) f_{b/B}(x_2, \mu) \frac{d\hat{\sigma}_{ab}(x_1, x_2, \mu)}{dX} \left[1 + \mathcal{O}\left(\frac{\Lambda_{\text{QCD}}^p}{Q^p}\right) \right]$$

Parton distribution functions
(long-distance, non-perturbative)
 $\pm(3 - 5) \%$

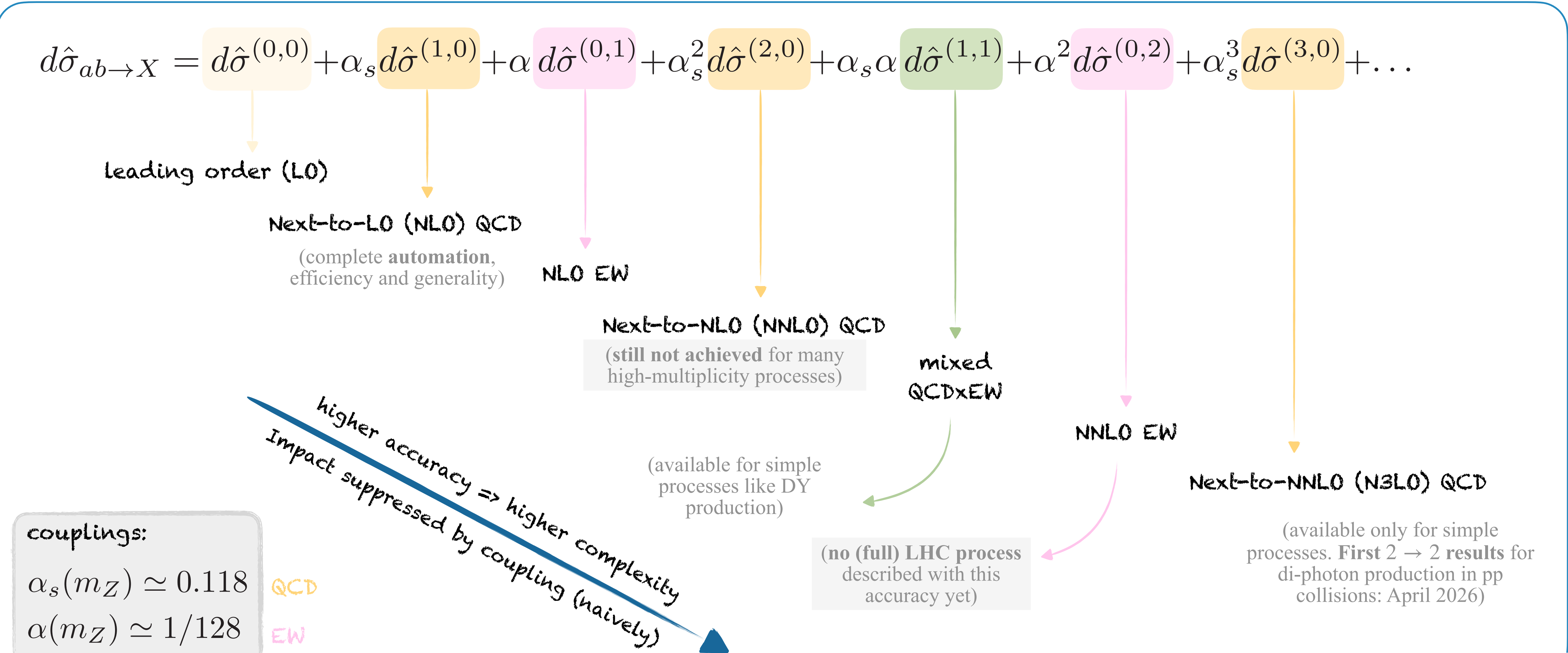
Hard scattering
(short-distance, perturbative regime)
Target:
 $\pm (\lesssim \text{a few percent})$

Non-perturbative effects
(Power corrections)
 $p \geq 1$

PRECISION&REALISM

At typical LHC energy scales: simultaneous expansion in QCD and EW couplings

energy current efforts
 $Q \gg \Lambda_{\text{QCD}}$ F.O. perturbation theory: N^xLO



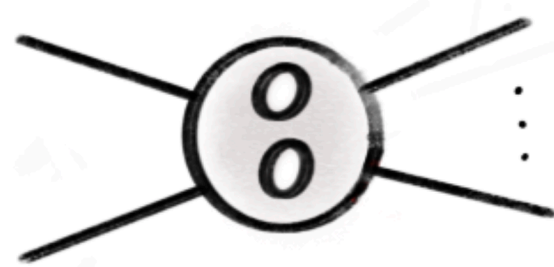
NNLO

NNLO QCD calculations are being extended to an increasing variety of processes

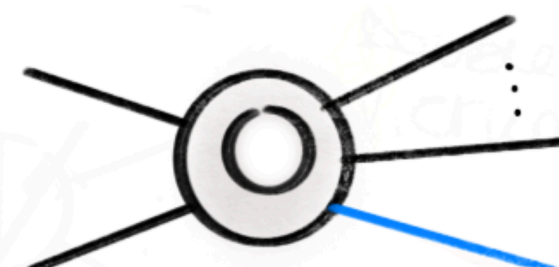
Adding external particles and/or additional mass scales remains a challenge

Schematic NNLO ingredients:

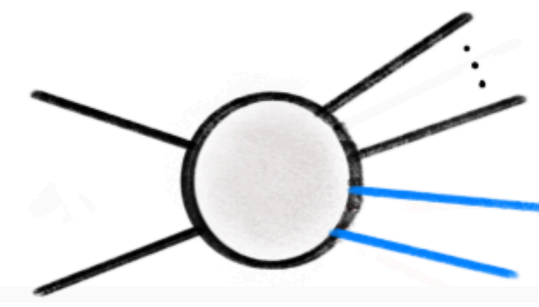
$$\frac{d\sigma_{\text{N}^2\text{LO}}}{dX} = \int d\Phi_n \textcolor{blue}{VV} \delta_{X_n} + \int d\Phi_{n+1} \textcolor{blue}{RV} \delta_{X_{n+1}} + \int d\Phi_{n+2} \textcolor{blue}{RR} \delta_{X_{n+2}}$$



Up to $1/\epsilon^4$
explicit poles



Up to $1/\epsilon^2$ explicit poles
Up to 2 PS singularities



up to 4 PS
singularities

$d = 4 - 2\epsilon$

Virtual corrections:

Algebraic and analytic complexity, increasing with the number of masses and external particles

Real radiation:

Systematic treatment of **soft and collinear singularities**
→ fully differential predictions

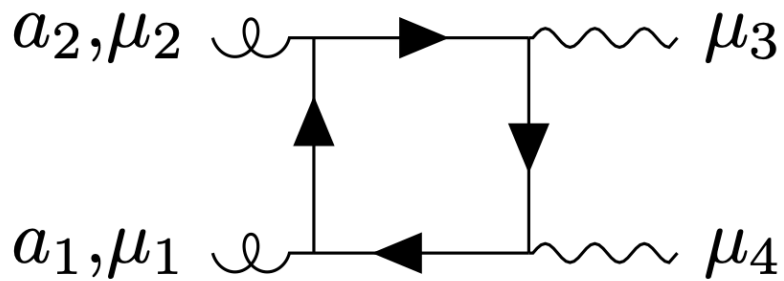
Ironically, in many cases the most complicated part, the double virtual, is also the “less relevant” (not always true!)



$X = \text{IRC-safe observable}$

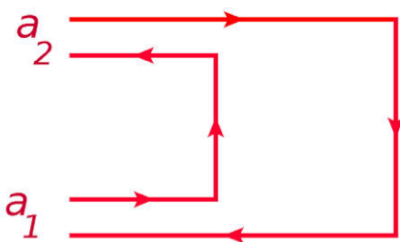
LOOP CORRECTIONS

- The **complexity increases** dramatically with **loop order, number of external particles and internal masses**
- **Beyond 1loop: still case-by-case calculation**
 - Today in QCD: 2loop $2 \rightarrow 3$ scattering with masses, 3loop $2 \rightarrow 2$ with one external mass
 - Today in EW: 2loop $2 \rightarrow 2$ with massive particles (Z, W, H, t) everywhere
- **Various approaches:** **analytic** (representation dependent), **numerical** (might be unstable), **approximate** (non universal)

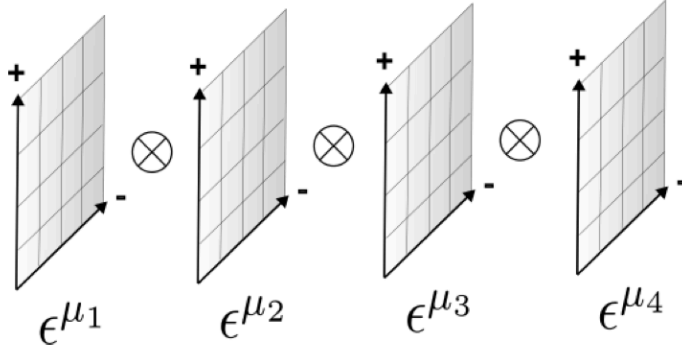


$$= g_s^4 \text{Tr}\{T^{a_1} T^{a_2}\} \int \frac{d^d k}{(2\pi)^d} \frac{\text{tr} \left[(\not{k}) \not{\epsilon}_1 (\not{k} + \not{p}_1) \not{\epsilon}_2 (\not{k} + \not{p}_{12}) \not{\epsilon}_3 (\not{k} + \not{p}_{123}) \not{\epsilon}_4 \right]}{(k)^2 (k + p_1)^2 (k + p_{12})^2 (k + p_{123})^2}$$

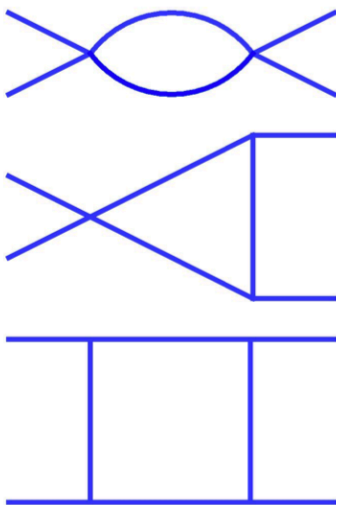
color



tensors



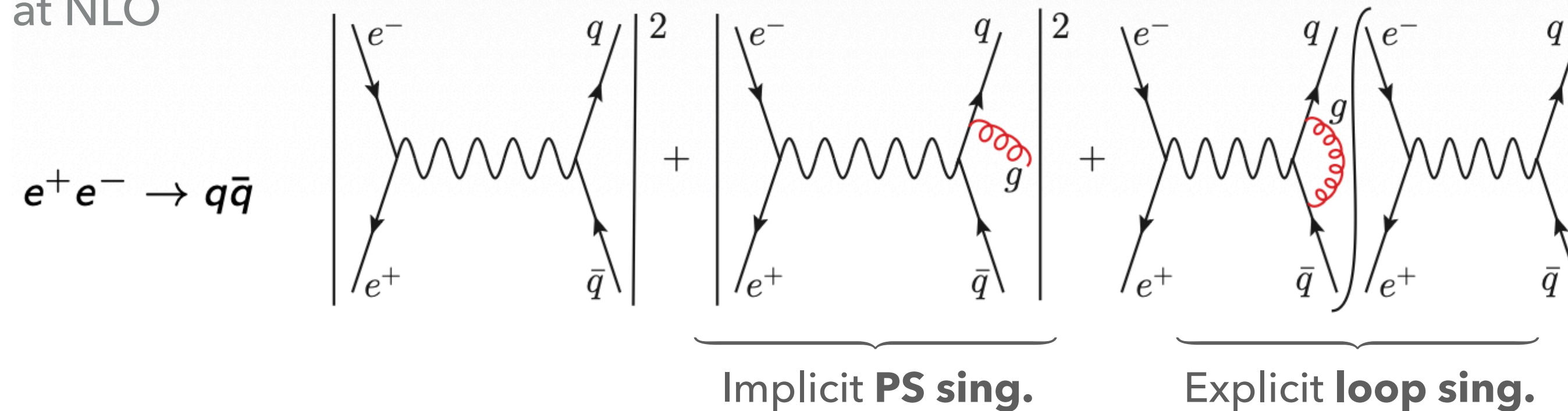
integrals



- **Algebraic complexity:** proliferation of scales
- **Analytic complexity:** new mathematical structures (special functions, elliptic and beyond)
- **Numerical complexity:** representations suitable for efficient and stable amplitude evaluation

REAL CORRECTIONS@NLO

- The problem emerges already at NLO



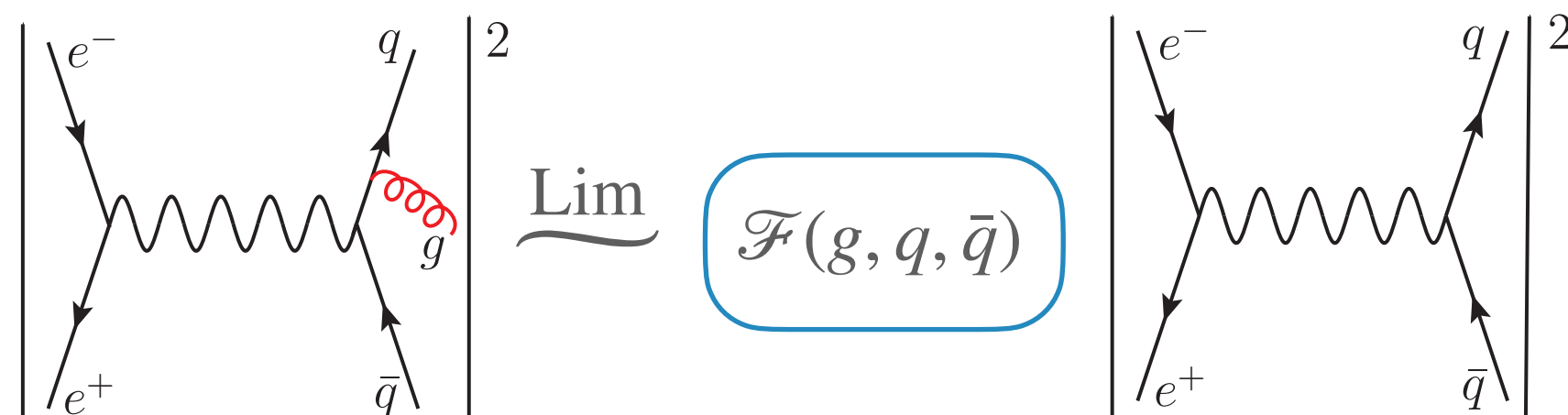
$$d = 4 - 2\epsilon$$

- The explicit poles of 1-loop amplitudes are known and universal

$$\mathcal{A}_{1\text{loop}} = \left[\sum_i \left(\frac{1}{\epsilon^2} + \frac{\gamma_i}{\mathbf{T}_i^2} \frac{1}{\epsilon} \right) \sum_{i \neq j} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \left(\frac{\mu^2}{-s_{ij}} \right)^\epsilon \right] \mathcal{A}_{\text{tree}} + \mathcal{A}_{1\text{loop}}^{\text{fin}}$$

Finite remainder requires **dedicated calculations on a process-by-process basis**. It enters directly **numerical implementations**

- Singular limits of tree-level amplitudes are known and universal



$\text{Lim} = E_g \rightarrow 0$: **soft limit** $\Rightarrow \mathcal{F}(g, q, \bar{q}) =$ eikonal function

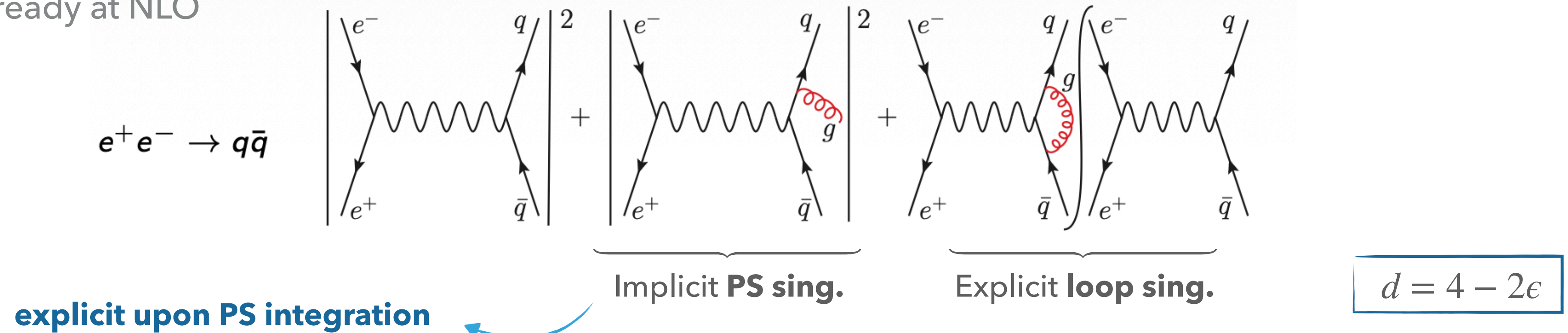
$\text{Lim} = \theta_{gq} \rightarrow 0$: **collinear limit** $\Rightarrow \mathcal{F}(g, q, \bar{q}) =$ splitting function

$\mathcal{F}(g, q, \bar{q})$ universal: depends only on external-parton momenta, spin, colour.

No explicit poles, unresolved PS integration generate $1/\epsilon$ poles.

REAL CORRECTIONS@NLO

- The problem emerges already at NLO



KLN

IR singularities vanish if all degenerated states with arbitrary multiplicities in the final state are combined
[Kinoshita '62; Lee, Nauenberg '64]

- However: need to **extract real-radiation IR poles without integrating** over PS → **retain fully differential predictions**

Subtraction:

conceptually non-trivial,
 but if **local** and **analytic**
 then extremely versatile
 and numerically stable

$$\int \text{diagram} d\Phi_g = \underbrace{\int \left[\text{diagram} - \text{Subtraction term} \right] d\Phi_g}_{\text{Finite in } d=4} + \underbrace{\int \text{diagram} d\Phi_g}_{1/\epsilon \text{ poles}}$$

Subtraction term

Integrated subtraction term

NLO

- Catani-Seymour (CS) [9602277]
- Frixione-Kunst-Signer (FKS) [9512328]

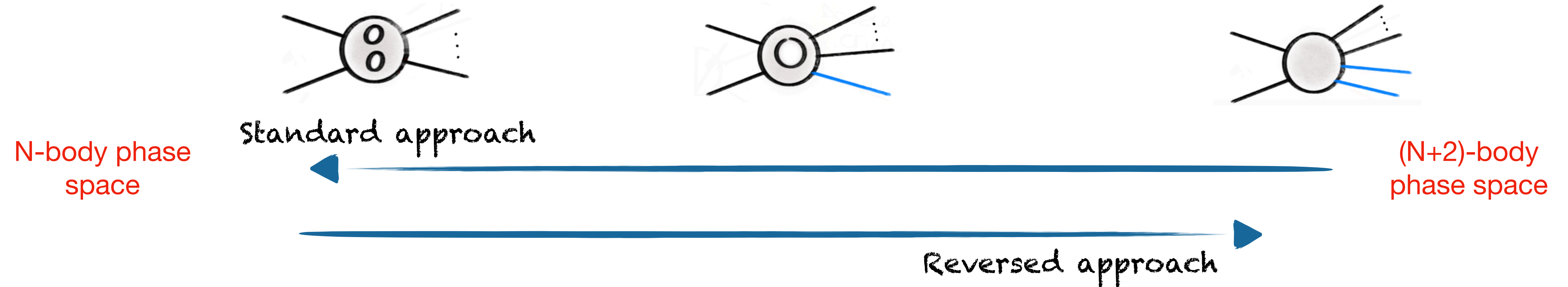
Currently implemented in full generality in fast and efficient NLO generators [HELAC-NLO, Rocket, BlackHat+SHERPA, GoSam+SHERPA/MADGRAPH, NJet+SHERPA, Madgraph5-aMC@NLO, RECOLA, OpenLoops+SHERPA...]

NNLO

- ▶ Numerous methods available
 - ▶ Antenna [Gehrmann-De Ridder et al. '05]
 - ▶ ColorfullNNLO [Del Duca et al. '16]
 - ▶ STRIPPER [Czakon '10]
 - ▶ Nested soft-collinear [Caola et al. '17]
 - ▶ Local analytic sector [Magnea, CSS et al. '18]
 - ▶ Geometric IR subtraction [Herzog '18]
 - ▶ Unsubtraction [Sborlini et al. '16]
 - ▶ FDR [Pittau '12]
 - ▶ Universal Factorisation [Sterman et al. '20]
- ▶ Some of them already applied to numerous pheno studies
- ▶ **None** of the existing subtraction schemes **can be considered "the ultimate"**

	NLO	NNLO	N ³ LO
▶ Given matrix elements and resources, N ^k LO corrections can in principle be computed	✓	✓ *	✗
▶ New N ^k LO calculations are actually streamlined	✓	✓ **	✗
▶ Different approaches/calculations available (validation, efficiency, ...)	✓	●	✗
▶ N ^k LO calculations can be easily reproduced and performed by anyone	✓	✗	✗
▶ N ^k LO event generation is possible	✓	✗	✗
* sector improved residue subtraction, antenna subtraction (massless), nested soft-collinear subtraction (massless), ... ** STRIPPER ✓ Yes ● Partial ✗ No			

NESTED SOFT-COLLINEAR SUBTRACTION: CHANGING THE PARADIGM



Idea: reverses the standard logic* and look at the pole structure of the virtual corrections to infer similar structures for the subtraction terms

[Devoto, Melnikov, Röntsch, CSS, Tagliabue '23]

- Introduce **universal operators** organising the counterterms, guided by the **form of the double virtual IR structure**
- Preserve as much as possible the **recurrences of NLO-like objects (and close variants)** in the finite remainder

(*) similar idea also explored in *antenna subtraction* [Gehrmann, Glover, Marcoli '23] and *local analytic sector subtraction* [Magnea, Milloy, CSS, Torrielli '24]

NESTED SOFT-COLLINEAR SUBTRACTION: NLO

$$2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = d\hat{\sigma}_{ab}^{\text{V}} + d\hat{\sigma}_{ab}^{\text{R}}$$

1) Virtual corrections:
color-correlations, elastic terms

$$I_{\text{V}}(\epsilon)$$

$\Rightarrow$

$$\bar{I}_1(\epsilon) = \frac{1}{2} \sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

2) Real corrections:

soft: color-correlations, elastic terms

$$I_{\text{S}}(\epsilon) = -\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j)$$

$$I_{\text{V}}(\epsilon) + I_{\text{S}}(\epsilon)$$

- Highest pole trivially cancels
- Singular color correlations cancel
- Remnant elastic single pole

hard-collinear: no color-correlations, elastic terms

$$I_{\text{C}}(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}$$

"generalised anomalous dimensions"

$$\Gamma_{i,f_i} = \gamma_i + 2\mathbf{T}_i^2 L_i + \mathcal{O}(\epsilon)$$

$\Rightarrow$

$$I_{\text{T}}(\epsilon) = I_{\text{V}}(\epsilon) + I_{\text{S}}(\epsilon) + I_{\text{C}}(\epsilon) \quad \text{FINITE}$$

3) Sum: $2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = \frac{\alpha_s(\mu)}{2\pi} \langle I_{\text{T}}^{(0)} \cdot F_{\text{LM}} \rangle + \langle F_{\text{LV}}^{\text{fin}} \rangle + \langle \mathcal{O}_{\text{NLO}} \Delta^{(\mathbf{m})} F_{\text{LM}}(\mathbf{m}) \rangle$

NESTED SOFT-COLLINEAR SUBTRACTION: SNAPSHOT OF NNLO

Similar guiding principle @NNLO:

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '24, 25]

- Rewrite the poles of VV in term of NLO operators:

Double virtual

$$\begin{aligned} \langle F_{VV} \rangle = & [\alpha_s]^2 \left\langle \left[\frac{1}{2} I_V^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \left(\frac{\beta_0}{\epsilon} I_V(\epsilon) - \left(\frac{\beta_0}{\epsilon} + K \right) I_V(2\epsilon) \right) \right] \cdot F_{LM} \right\rangle \\ & + [\alpha_s]^2 \left\langle \left[-\frac{1}{2} [\bar{I}_1(\epsilon), \bar{I}_1^\dagger(\epsilon)] + \mathcal{H}_{2,tc} + \mathcal{H}_{2,tc}^\dagger + \mathcal{H}_{2,cd} + \mathcal{H}_{2,cd}^\dagger \right] \cdot F_{LM} \right\rangle + [\alpha_s] \langle I_V(\epsilon) \cdot F_{LV}^{\text{fin}} \rangle + \dots \end{aligned}$$

- Find analogous structure in the subtraction terms of RR

Double soft

Factorised term

$$(T_i \cdot T_j) \cdot (T_k \cdot T_l)$$

$\Rightarrow$

$$\langle S_{mn} \Theta_{mn} F_{LM}(\mathbf{m}, \mathbf{n}) \rangle_{T^4} = [\alpha_s]^2 \frac{1}{2} \langle I_S^2(\epsilon) \cdot F_{LM} \rangle$$

- And in those of RV

Soft real-virtual

$$\langle S_m F_{RV}(\mathbf{m}) \rangle = [\alpha_s]^2 \left\langle \frac{1}{2} [I_S(\epsilon) \cdot I_V(\epsilon) + I_V(\epsilon) \cdot I_S(\epsilon)] \cdot F_{LM} \right\rangle + \dots$$

- Finally, look at the collinear component of RR (combine sectors at a symbolic level before integrating)

Double collinear

$$\frac{1}{2} \sum_{i,j=1}^{N_p} \langle \bar{S}_m \bar{S}_n C_{im} C_{jn} \Delta^{(mn)} F_{LM}(\mathbf{m}, \mathbf{n}) \rangle = [\alpha_s]^2 \left\{ \frac{1}{2} \langle I_C^2(\epsilon) \cdot F_{LM} \rangle + \dots \right.$$

NESTED SOFT-COLLINEAR SUBTRACTION: SNAPSHOT OF NNLO

Similar guiding principle @NNLO:

- Rewrite the poles of VV in term of NLO operators:

Double virtual

- Find analogous structure in the subtraction terms of RR

Double soft

- And in those of RV

Soft real-virtual

- Finally, look at the collinear component of RR

Double collinear

$$d\hat{\sigma}^{\text{NNLO}} = d\hat{\sigma}^{\text{VV}} + d\hat{\sigma}^{\text{RV}} + d\hat{\sigma}^{\text{RR}} + d\hat{\sigma}^{\text{pdf}}$$

Double-Virtual
Real-Virtual
Double-Real
PDFs Renor.

$$= \frac{[\alpha_s]^2}{2} \langle M_0 | [I_V + I_S + I_C]^2 | M_0 \rangle + \dots$$

$$\equiv \langle M_0 | I_T^2 | M_0 \rangle + \dots$$

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = \boxed{d\hat{\sigma}_{ab}^{\text{SU}}} + \boxed{d\hat{\sigma}_{ab}^{\text{DU}}} + \boxed{d\hat{\sigma}_{ab}^{\text{FR}}}$$

double-virtual real-virtual double-real coll. renor.

single-unr. double-unr. fully-resolved

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]

fully-resolved = (N+2)-body kinematics
 single-unresolved = (N+1)-body kinematics
 double-unresolved = N-body kinematics

- ❖ Formulas are compact
- ❖ Very few inputs for actual implementation
- ❖ Modular structure
- ❖ Easy to adapt to lepton-lepton and lepton-hadron collisions

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} =$$

double-virtual real-virtual double-real coll. renor.

$$= d\hat{\sigma}_{ab}^{\text{SU}} + d\hat{\sigma}_{ab}^{\text{DU}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

single-unr. double-unr. fully-resolved

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]

fully-resolved = (N+2)-body kinematics
 single-unresolved = (N+1)-body kinematics
 double-unresolved = N-body kinematics

$$d\hat{\sigma}_{ab}^{\text{FR}} = \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \Theta_{mn} \mathbb{F}_{\text{LM}}^{ab, \text{DS}}[\mathbf{m}, \mathbf{n}] \rangle + \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \mathbb{F}_{\text{LM}}^{ab, \text{DS}}[\mathbf{m}, \mathbf{n}] \rangle$$

$$d\hat{\sigma}_{ab}^{\text{SU, sb, a}} = [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(a, \mathbf{m})} \omega_{a||\mathbf{n}}^{ma, na} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{am}}{2}\right) \hat{P}_{xa}^{(0)} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\ \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right\},$$

$$d\hat{\sigma}_{ab}^{\text{SU, sb, b}} = [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(b, \mathbf{m})} \omega_{b||\mathbf{n}}^{mb, nb} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{bm}}{2}\right) F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \hat{P}_{xb}^{(0)} \rangle \right. \\ \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle \right\}.$$

$$d\hat{\sigma}_{ab}^{\text{SU, el}} = \sum_{\mathbf{m}} \sum_n \left\{ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\mathbf{m}}) + \bar{\delta}_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\text{max}})] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\ - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{i||\mathbf{n}}^{mi, ni} \log\left(\frac{\eta_{im}}{2}\right) \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} (\gamma_i + 2T_i^2 L_i(E_{\mathbf{m}})) + \bar{\delta}_{\mathbf{m}g} (\gamma_i + 2T_i^2 L_i)] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\ - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{mi, ni} \log\left(\frac{\eta_{im}}{4(1-\eta_{im})}\right) \Delta^{(\mathbf{m})} [\gamma_{\mathbf{m}} + 2T_q^2 L_{\mathbf{m}} \bar{\delta}_{\mathbf{m}g}] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\ + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{RV, fin}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \Big\} + \sum_n \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{mi, ni} \Delta^{(\mathbf{m})} [\gamma_g^{\perp, r} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \\ + \gamma_g^{\perp} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) F_{\text{LM, } \mu\nu}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \rangle \Big\}.$$

$$d\hat{\sigma}_{ab}^{\text{DU, db}} = \sum_{x, y} \sum_n [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xy}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{yb}^{\text{NLO}} \rangle$$

$$d\hat{\sigma}_{ab}^{\text{DU, sb, a}} = \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes \mathcal{F}^{xb}[\mathcal{B}_{N, n}] \rangle + [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NNLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N, n}] \rangle \right] \\ + \sum_{n'} [\alpha_s]^2 \langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n'}]] \rangle$$

$$d\hat{\sigma}_{ab}^{\text{DU, sb, b}} = \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{F}^{ax}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle + [\alpha_s]^2 \langle F_{\text{LM}}^{ax}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{xb}^{\text{NNLO}} \rangle \right] \\ + \sum_{n'} [\alpha_s]^2 \langle [\mathcal{W}_b^{b||\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n'}]] \otimes \mathcal{P}_{bb}^{\mathcal{W}} \rangle$$

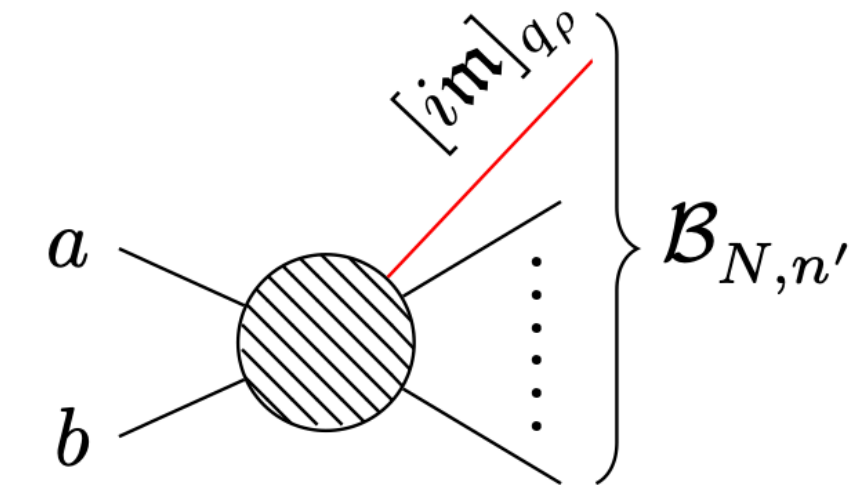
$$d\hat{\sigma}_{ab}^{\text{DU, el}} = \sum_n \left\{ [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n}] \rangle \right. \\ + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \langle [\theta_{\mathcal{H}_f} \gamma_{z, f_i \rightarrow f_i g}^{\mathcal{W}}(L_i) \mathcal{W}_i^{i||\mathbf{n}, \text{fin}} + \delta^{(0)} \mathcal{W}_i^{\mathbf{m}||\mathbf{n}, \text{fin}} + \delta^{\perp, (0)} \mathcal{W}_r^{(i)}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n}] \rangle \\ + [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LV, fin}}^{ab}[\mathcal{B}_{N, n}] \rangle + \langle F_{\text{LV}^2, \text{fin}}^{ab}[\mathcal{B}_{N, n}] \rangle + \langle F_{\text{VV, fin}}^{ab}[\mathcal{B}_{N, n}] \rangle \Big\}$$

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = \boxed{d\hat{\sigma}_{ab}^{\text{SU}}} + \boxed{d\hat{\sigma}_{ab}^{\text{DU}}} + \boxed{d\hat{\sigma}_{ab}^{\text{FR}}}$$

double-virtual
real-virtual
double-real
coll. renor.

single-unr.
double-unr.
fully-resolved

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]



Very few inputs for actual implementation

- List of N , $N+1$ and $N+2$ external particles $\rightarrow$ select ingredients
- Born-level configurations compatible with the given theory/process
- 1-loop, 2-loop N -point finite remainders $\rightarrow$ dedicated calculation needed
- details (partitions, azimuthal direction, etc) $\rightarrow$ optimise for a given process

1. Fix initial-state partons (ab)
2. Sum over all partons x (y) compatible with 1.
3. Sum over all Born-level final-state configurations n compatible with 1. and 2.
 $\rightarrow$ this can be **parallelised**!
4. Universal ingredients (provided in ancillary files) need to be coded once for all

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{DU,db}} &= \sum_{x,y} \sum_n [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xy}[\mathcal{B}_{N,n}] \otimes \mathcal{P}_{yb}^{\text{NLO}} \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU,sb,a}} &= \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes \mathcal{F}^{xb}[\mathcal{B}_{N,n}] \rangle + [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NNLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N,n}] \rangle \right] \\
 &\quad + \sum_{n'} [\alpha_s]^2 \langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||n,\text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n'}]] \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU,sb,b}} &= \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{F}^{ax}[\mathcal{B}_{N,n}] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle + [\alpha_s]^2 \langle F_{\text{LM}}^{ax}[\mathcal{B}_{N,n}] \otimes \mathcal{P}_{xb}^{\text{NNLO}} \rangle \right] \\
 &\quad + \sum_{n'} [\alpha_s]^2 \langle [\mathcal{W}_b^{b||n,\text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n'}]] \otimes \mathcal{P}_{bb}^{\mathcal{W}} \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU,el}} &= \sum_n \left\{ [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \rangle \right. \\
 &\quad + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \langle [\theta_{\mathcal{H}_f} \gamma_{z,f_i \rightarrow f_{ig}}^{\mathcal{W}}(L_i) \mathcal{W}_i^{i||n,\text{fin}} + \delta^{(0)} \mathcal{W}_i^{m||n,\text{fin}} + \delta^{\perp,(0)} \mathcal{W}_r^{(i)}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \rangle \\
 &\quad \left. + [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LV,fin}}^{ab}[\mathcal{B}_{N,n}] \rangle + \langle F_{\text{LV}^2,\text{fin}}^{ab}[\mathcal{B}_{N,n}] \rangle + \langle F_{\text{VV,fin}}^{ab}[\mathcal{B}_{N,n}] \rangle \right\}
 \end{aligned}$$



PHENOMENOLOGICAL APPLICATIONS

MIXED QCDXEW CORRECTIONS

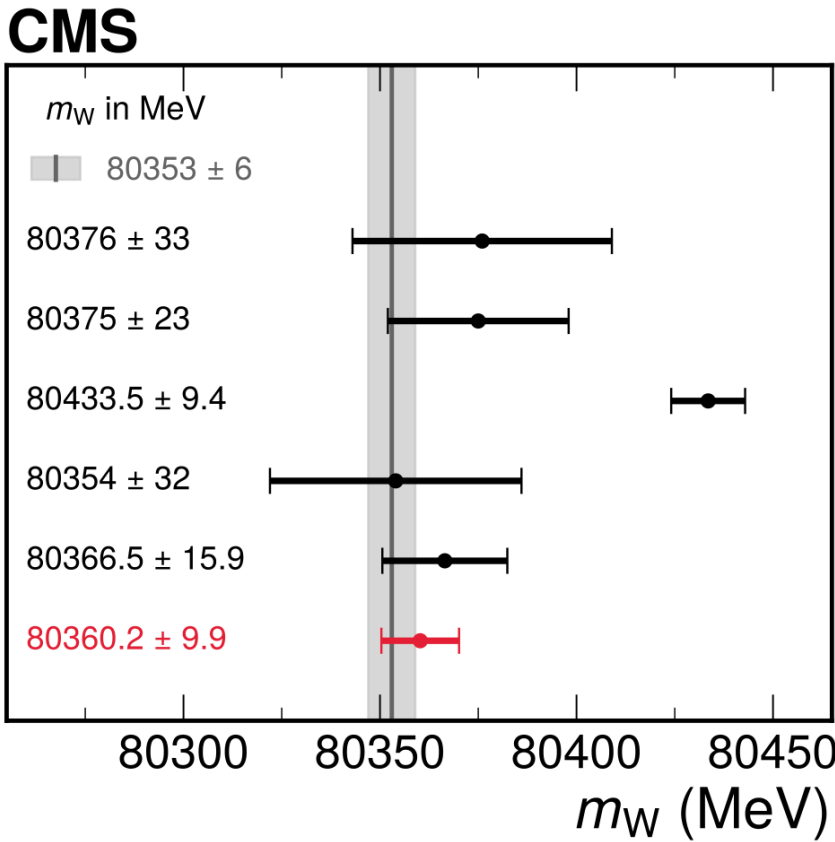
► Mixed QCD×EW corrections to the Drell-Yan process : increase our control on SM parameters and search for new physics

Resonance region:
EW parameters determination

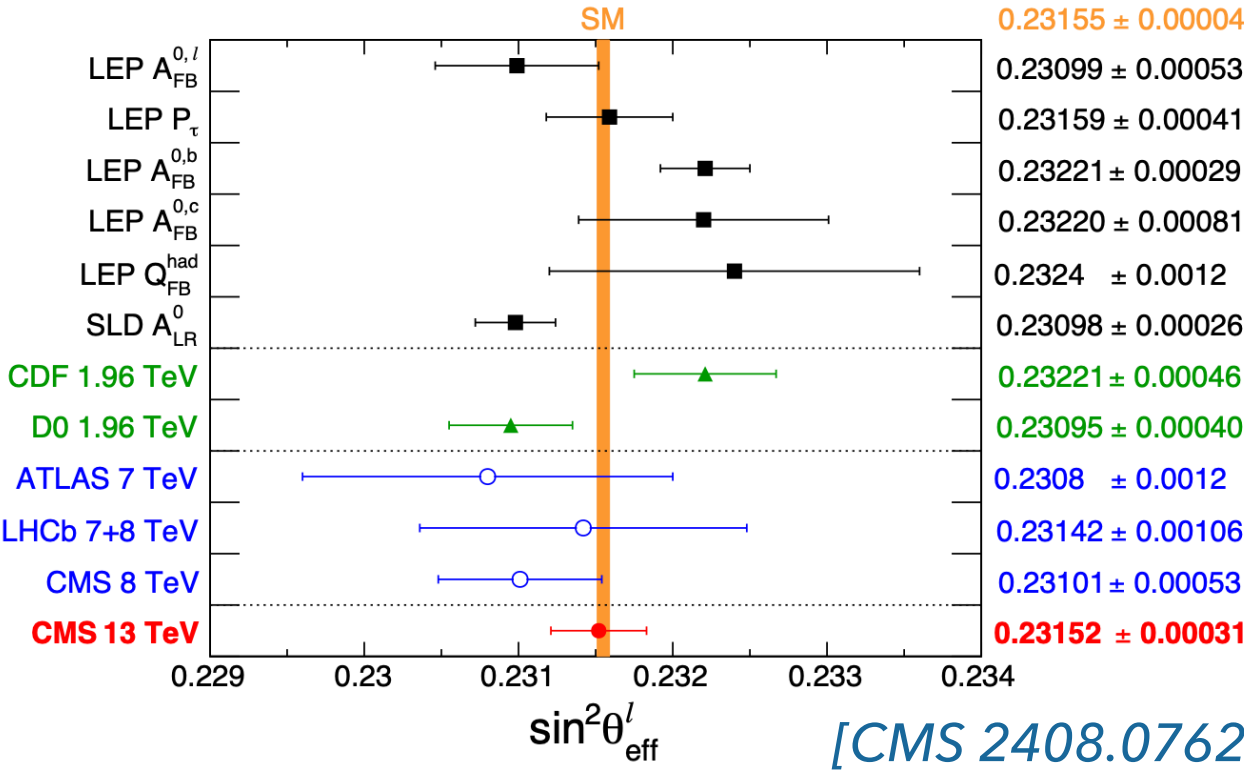
► m_W

Electroweak fit
PRD 110 (2024) 030001
LEP combination
Phys. Rep. 532 (2013) 119
D0
PRL 108 (2012) 151804
CDF
Science 376 (2022) 6589
LHCb
JHEP 01 (2022) 036
ATLAS
arXiv:2403.15085
CMS
This work

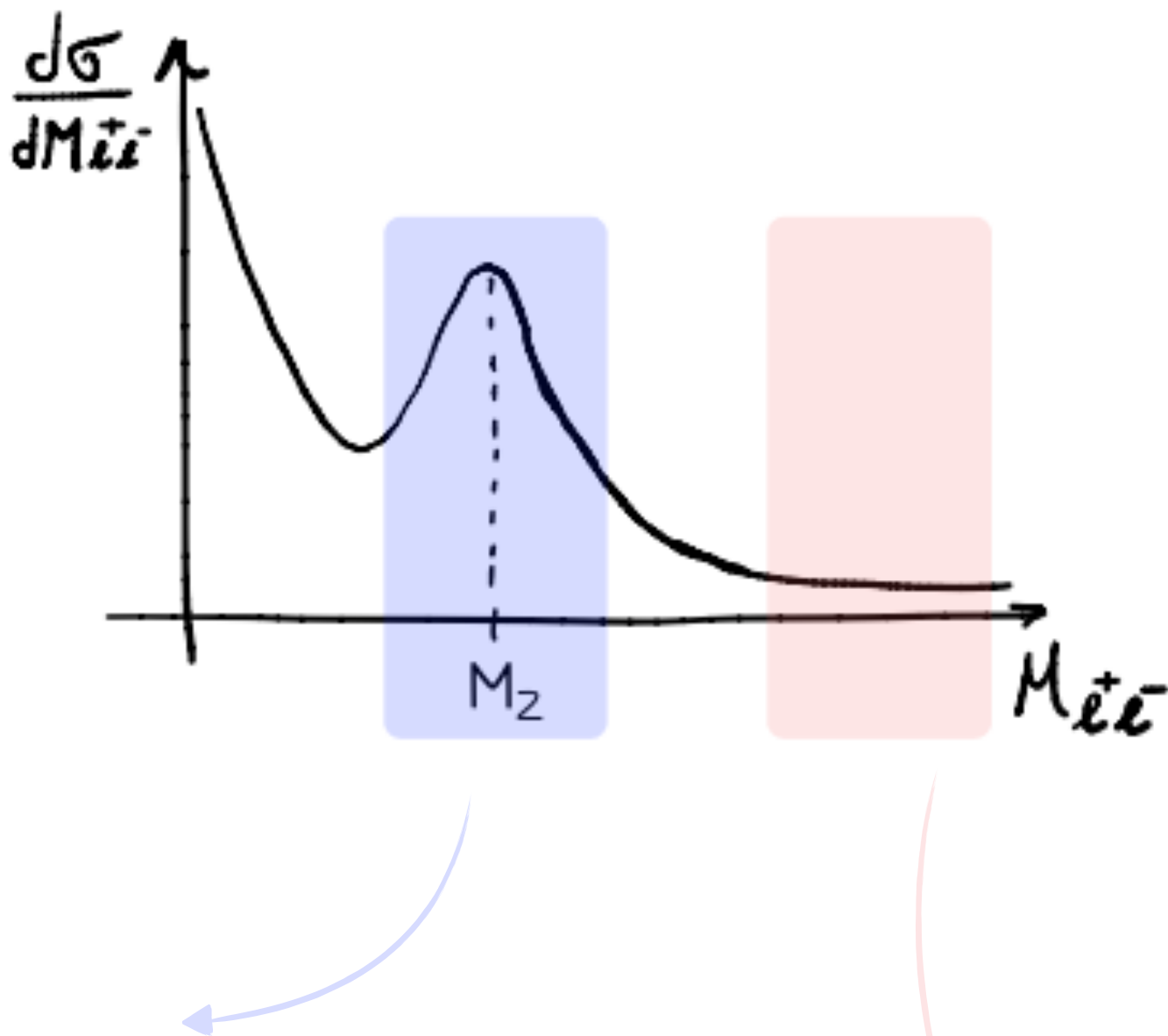
[2412.13872]



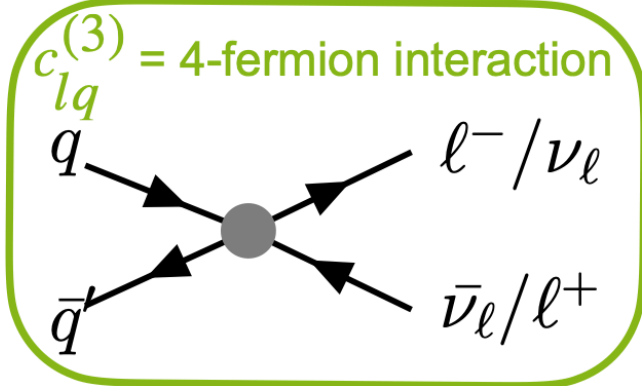
► $\sin^2 \theta_W$



[CMS 2408.07622]

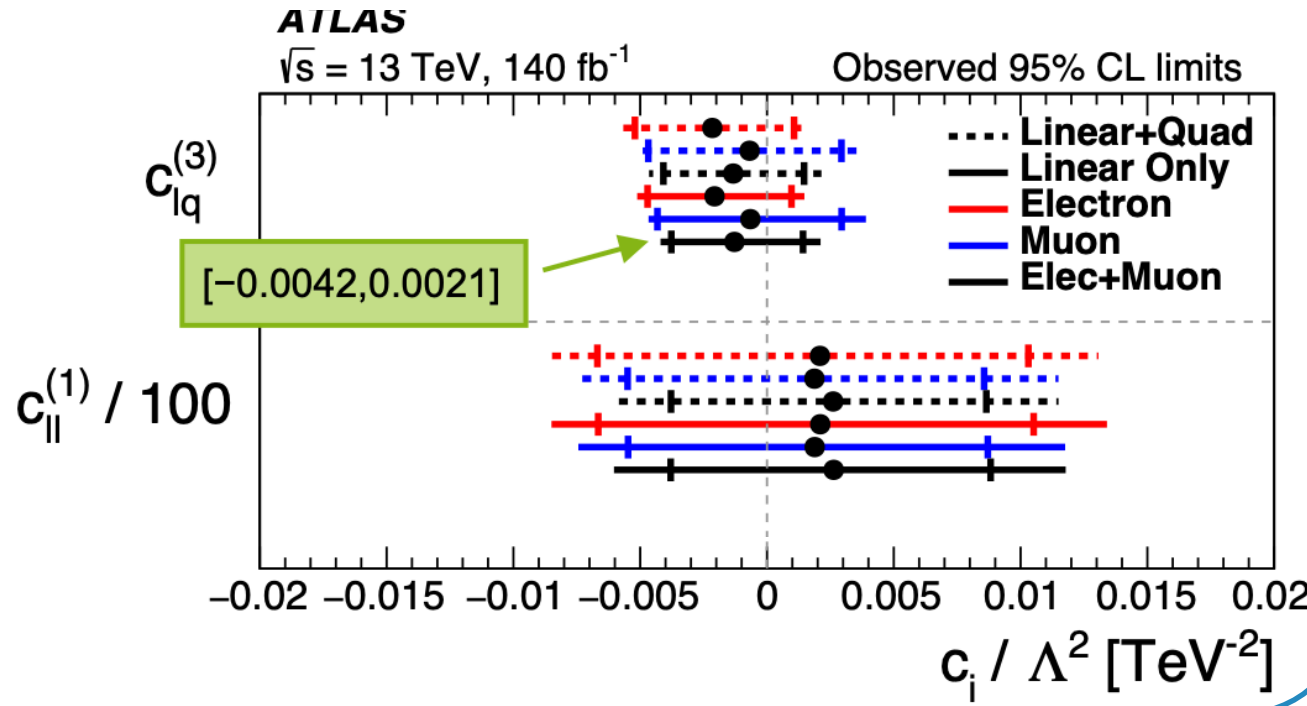
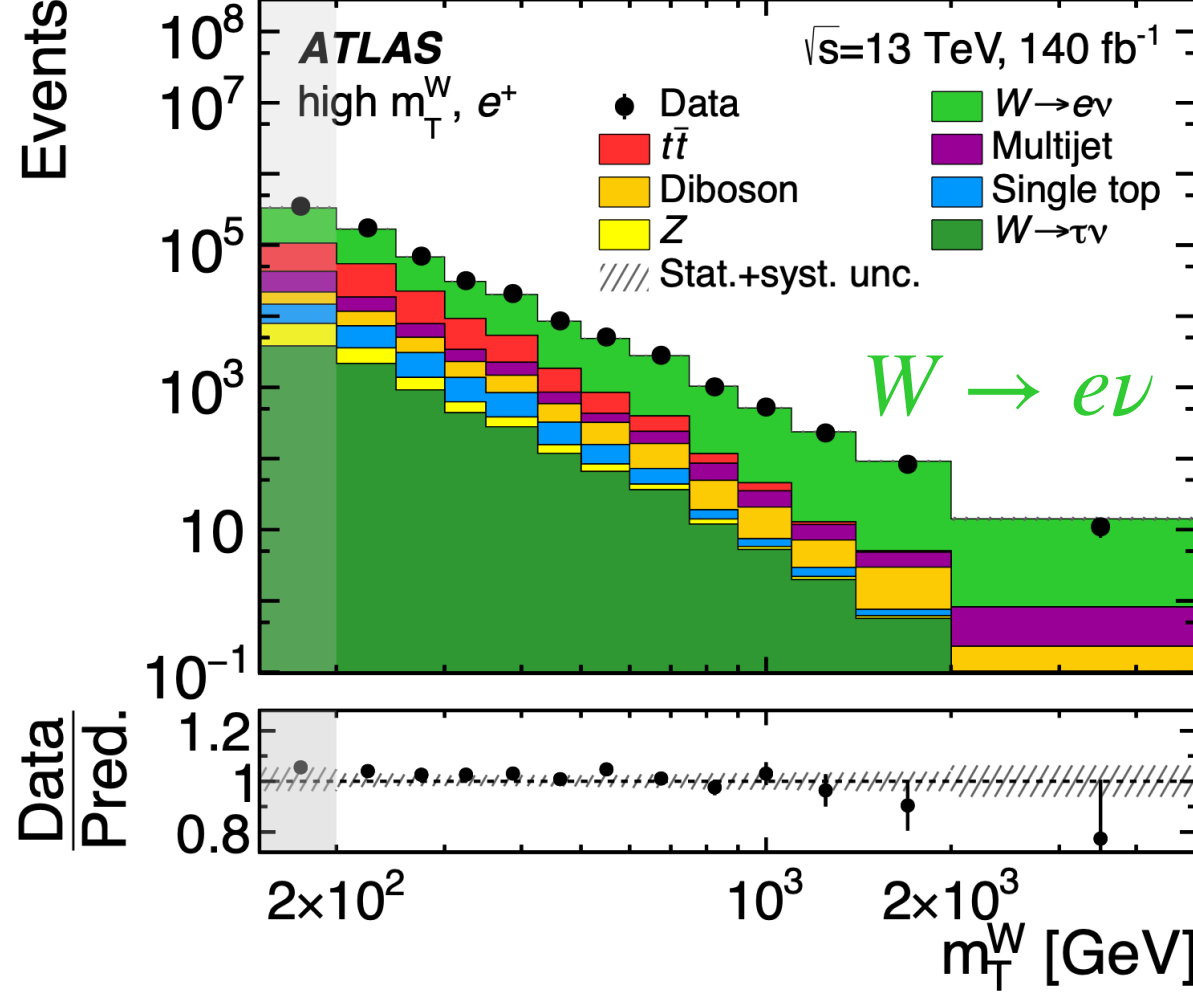


Constrain heavy NP in a model-independent way using **SMEFT**



Off-shell region:
New physics searches

[2502.21088]



MIXED QCD×EW CORRECTIONS

$$d\hat{\sigma}_{ab \rightarrow X} = d\hat{\sigma}^{(0,0)} + \alpha_s d\hat{\sigma}^{(1,0)} + \alpha d\hat{\sigma}^{(0,1)} + \alpha_s^2 d\hat{\sigma}^{(2,0)} + \alpha_s \alpha d\hat{\sigma}^{(1,1)} + \alpha^2 d\hat{\sigma}^{(0,2)} + \alpha_s^3 d\hat{\sigma}^{(3,0)} + \dots$$

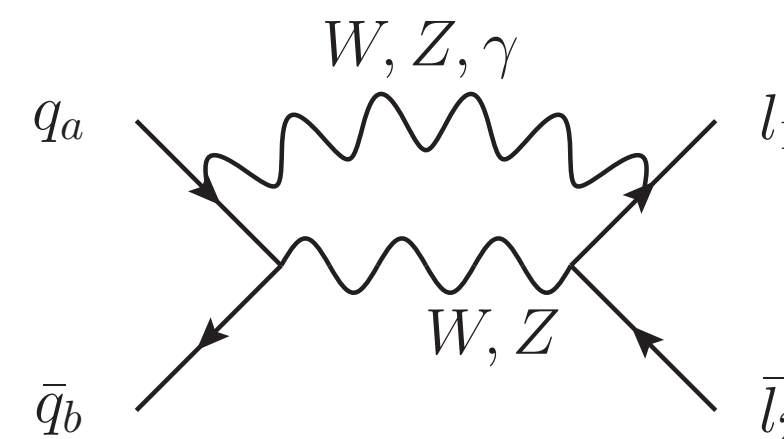
Couplings: $\alpha_s \sim 0.1$, $\alpha_{ew} \sim 0.01$

Mixed QCD×EW corrections:
 $\sim \mathcal{O}(\alpha_s \alpha_{ew}) \sim 0.1 \%$

At high energy, this naive power counting is spoiled by **Sudakov logarithms**:

in EW theory $m_{W/Z}$ provides a physical cut-off: real Z, W can be detected as distinguishable particles.

Large logs from virtual corrections are of physical significance and **spoil the naive power counting** [Kuhn, Penin, Smirnov '00][Ciafaloni, Ciafaloni, Comelli '01][Denner, Pozzorini '01]



$$\sqrt{s} = 1\text{TeV} \gg m_W$$

$$\frac{\alpha_{ew}}{4\pi \sin^2 \theta_W} \log^2 \left(\frac{s}{m_W^2} \right) \sim 6 \%$$

$$\frac{\alpha_{ew}}{4\pi \sin^2 \theta_W} \log \left(\frac{s}{m_W^2} \right) \sim 1 \%$$

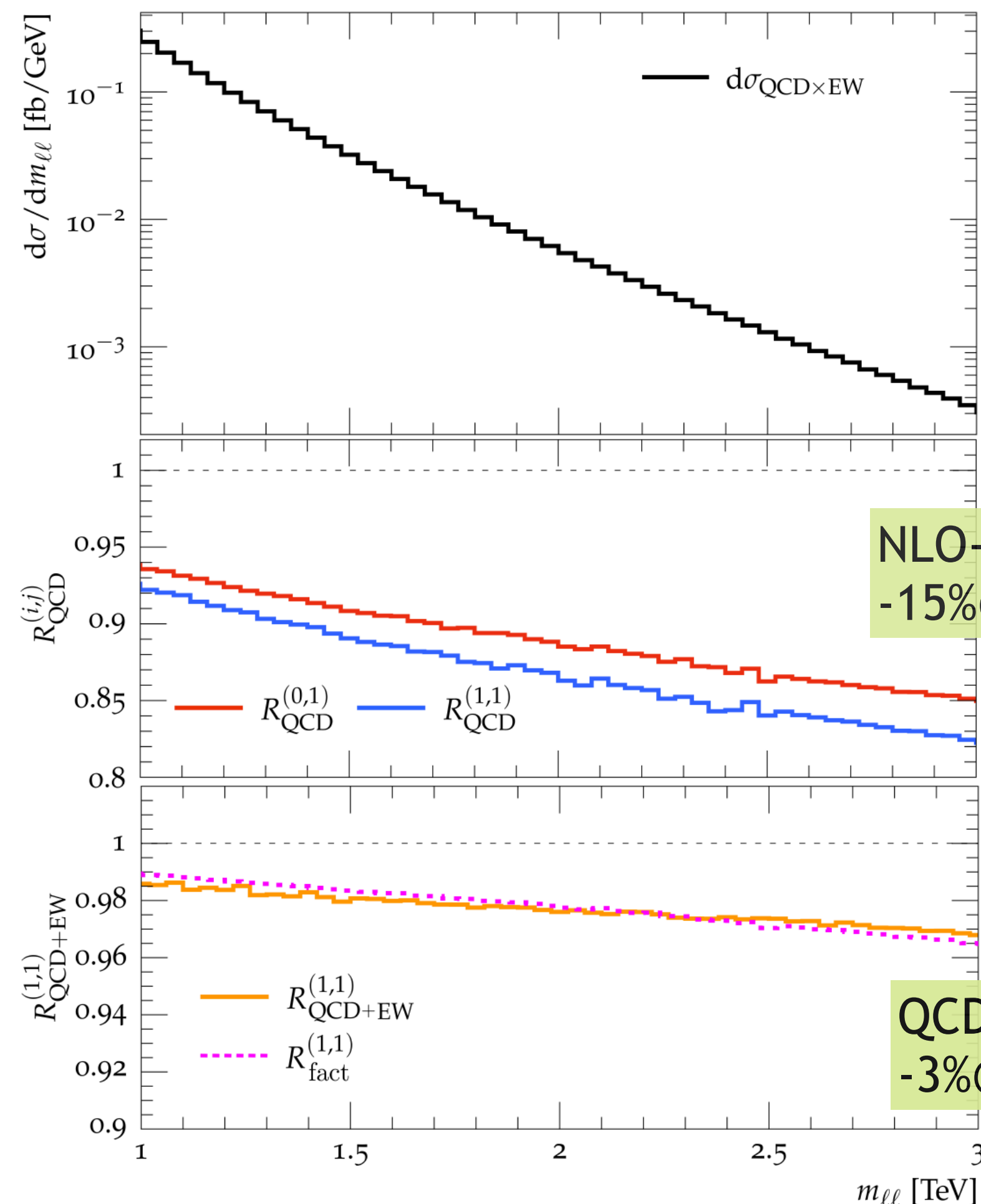
More accurate expectation:
(20% QCD) × (10% EW)

Potential **2%** correction in the high mass region

MIXED QCDxEW CORRECTIONS

Neutral current DY [Buccioni, [CSS](#), et al. '22]

Inclusive: QCDxEW $\sim -1\%$ LO,
NNLO QCD $\sim +0.8\%$ LO

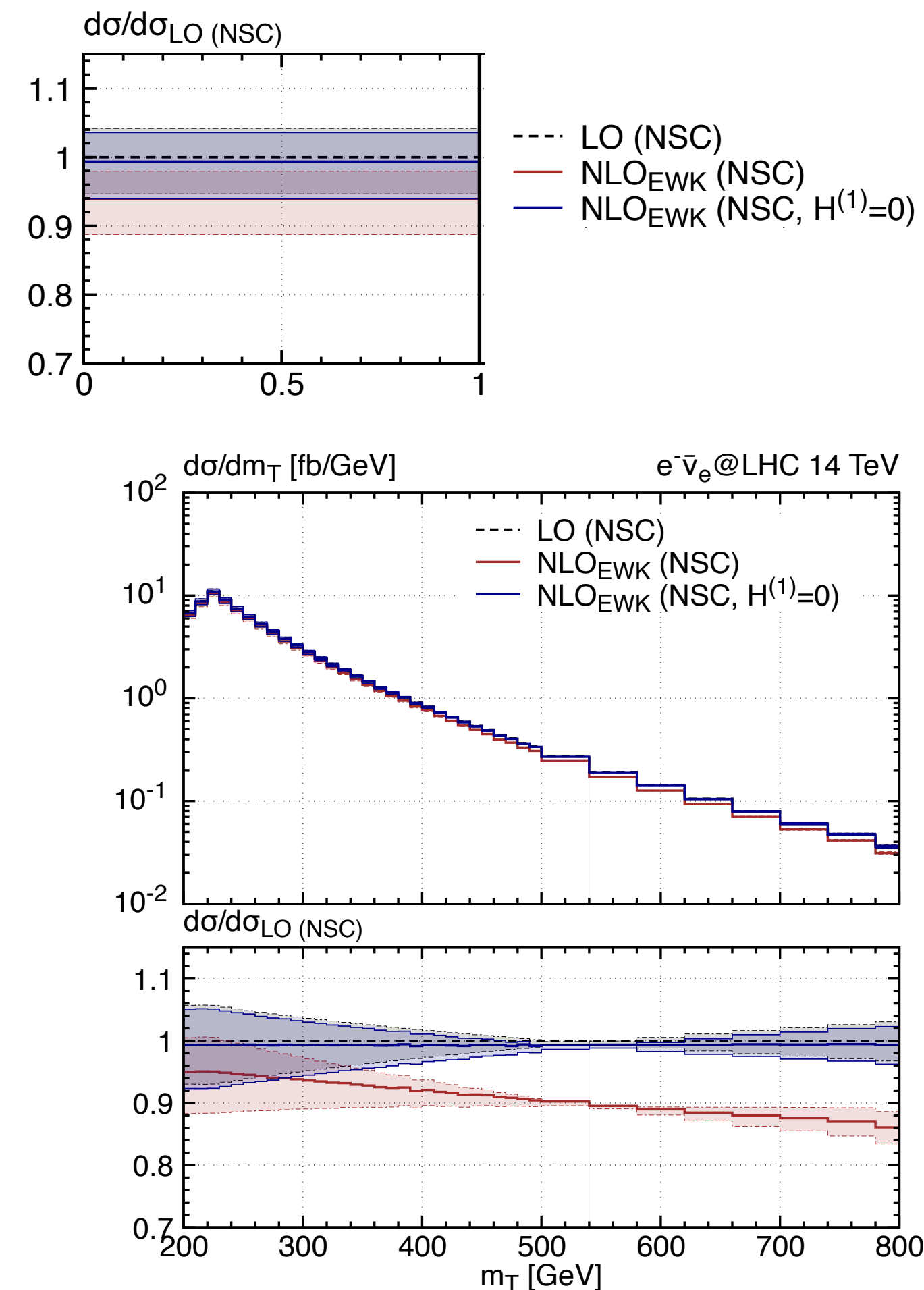


NLO-EW:
-15% @ 3 TeV

QCDxEW:
-3% @ 3 TeV

Charge current DY

[Biello, [CSS](#), et al. *in progress*]



$H(1)$ = finite part of 1loop squared amplitudes

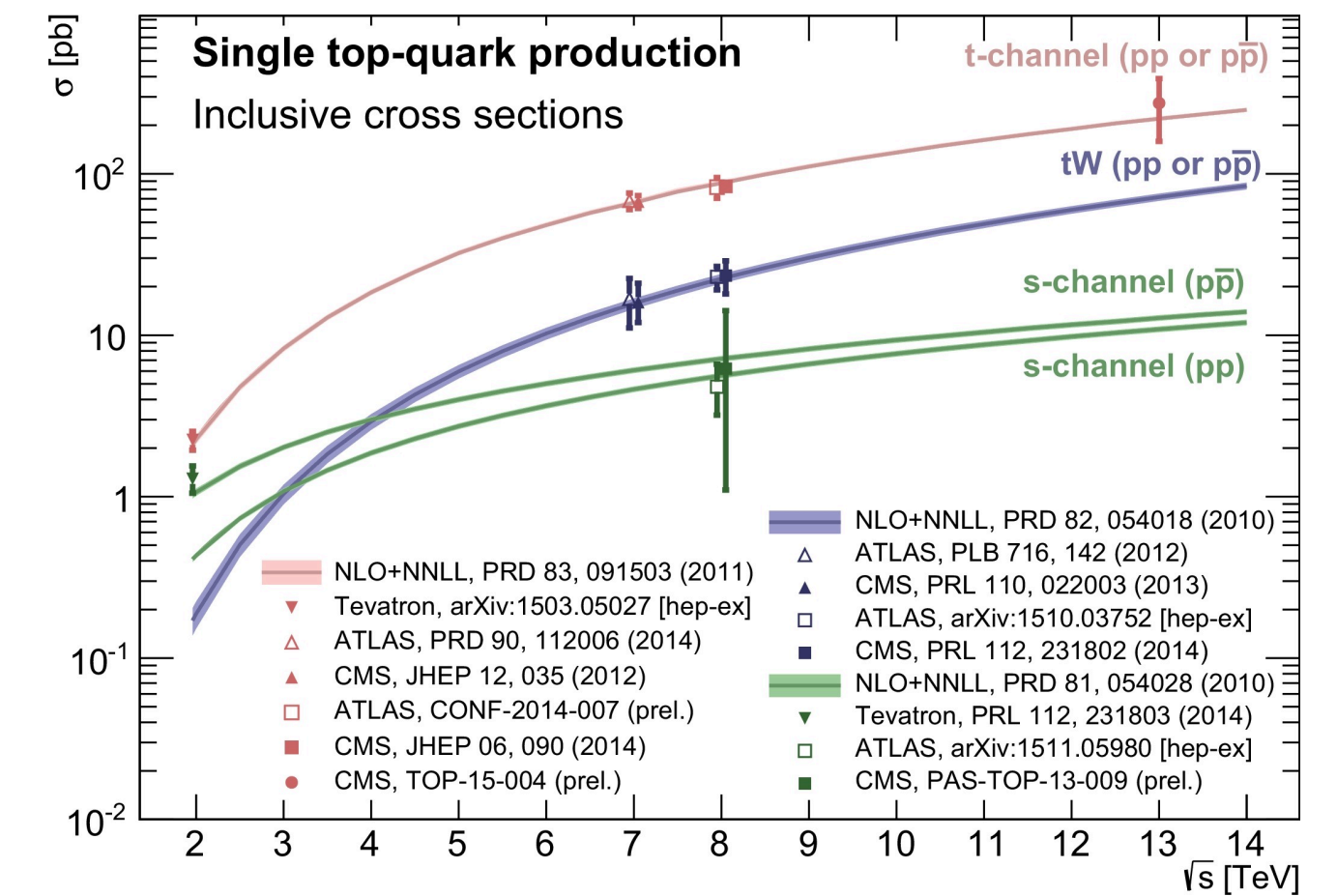
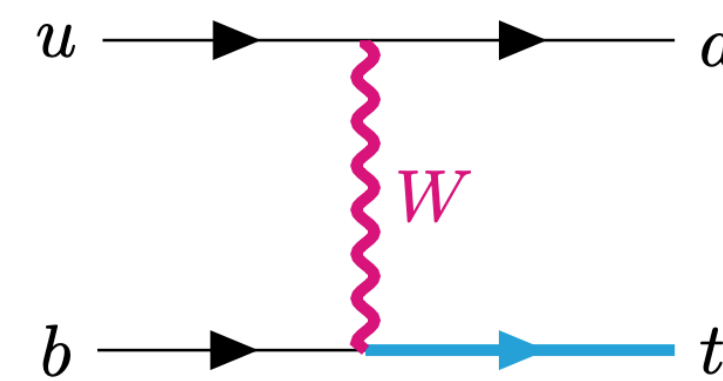
Beyond DY?

[Biello, [CSS](#), et al. *in progress*]

- ▶ Mixed correction calculations still proceed on a process-by-process base.
- ▶ The increasing experimental interest calls for a **systematic framework**
- ▶ Fully general final formulas for mixed QCDxEW subtraction are within sight [[Loops&Legs2026](#)]
- ▶ The idea is to “**abelianise**” the results recently obtained for **arbitrary QCD@NNLO** processes at collider [[Devoto, \[CSS\]\(#\) et al.](#)]

SINGLE TOP PRODUCTION

- Top quark is the heaviest particle of SM $\rightarrow$ substantial Yukawa coupling (special relation with Higgs boson)
- LHC as top-quark factory:** 25% of the top quarks are produced via "single top" production: t -channel accounts for $\sim 70\% \sigma_t$
- Electroweak mediated**
 - Determination of the **CKM matrix** V_{bt}
 - Indirect determination of **width** Γ_t and **mass** m_t
 - Bottom-quark PDF



Factorisable contributions

- Structure function approximation

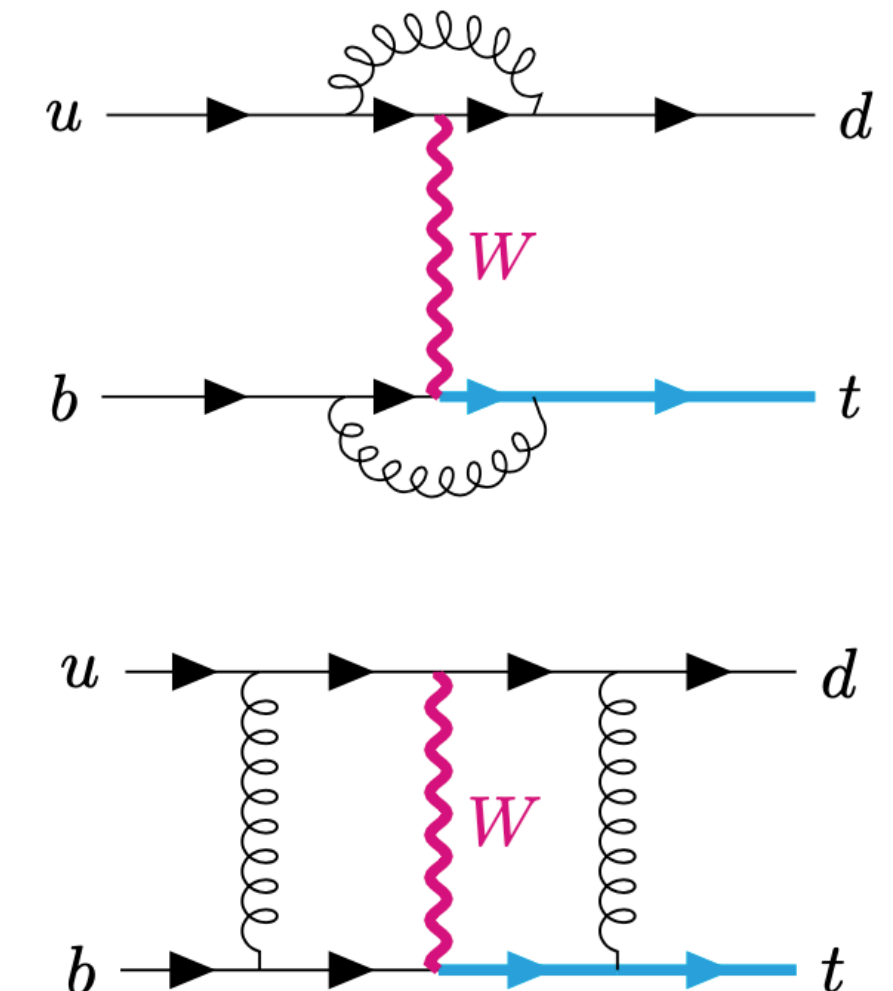
[Campbell, Neumann, Sullivan '21]

$$\mu_R = \mu_F = m_t = 172.5 \text{ GeV.}$$

$$\text{@14TeV: } \delta\sigma^{\text{NNLO}} \sim -0.7\% \sigma^{\text{NLO}}$$

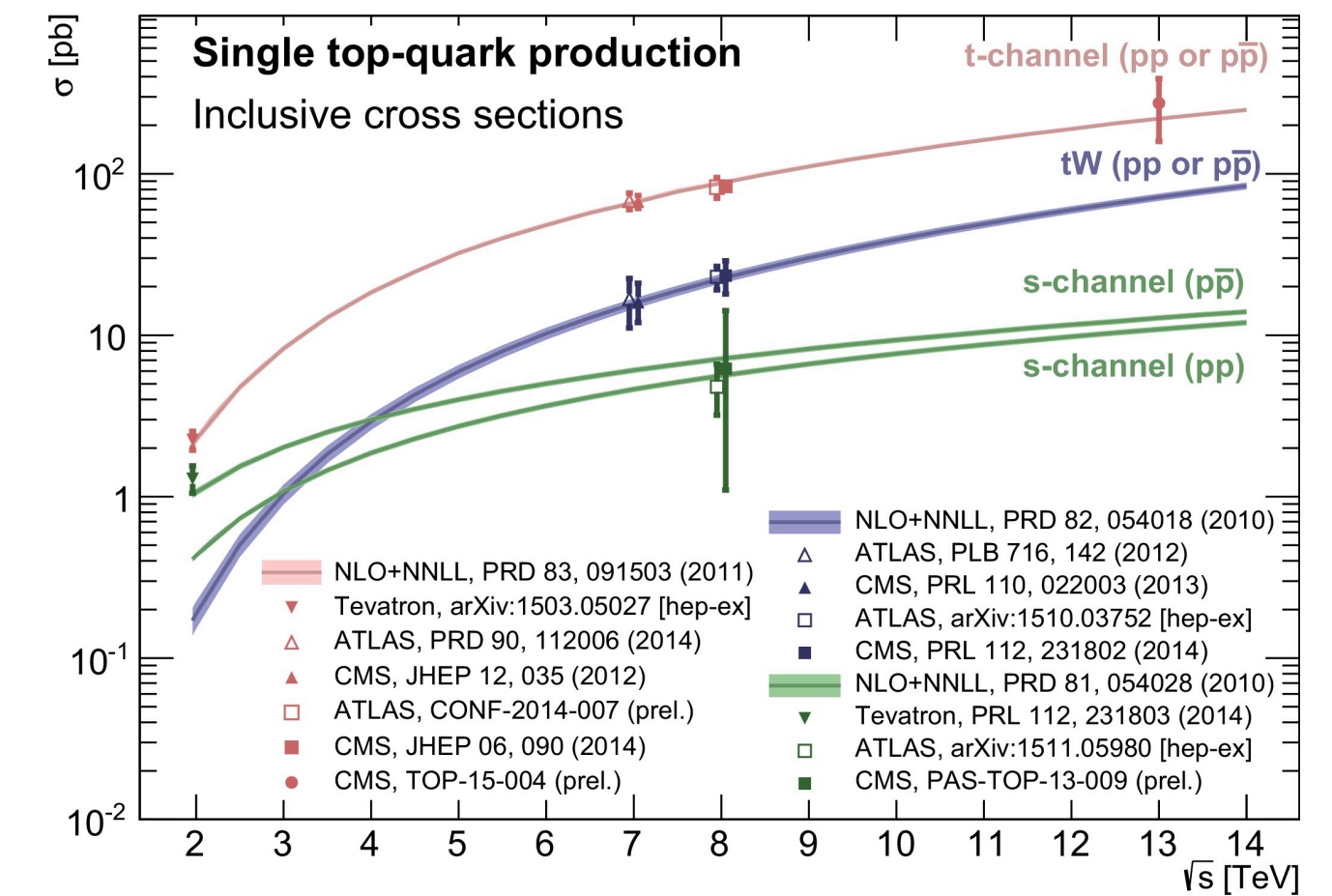
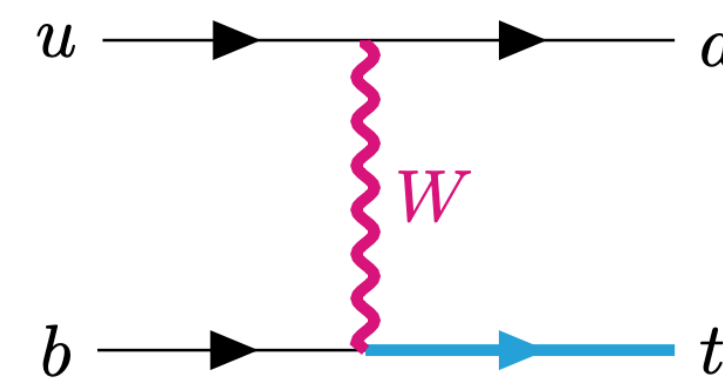
Non-factorisable contributions

- **Vanish at NLO** $\rightarrow$ magnitude of NNLO corrections cannot be inferred from scale variation, nor does it indicate the optimal scale choice
- **Colour suppressed** by $N_c^2 - 1 = 8$ @NNLO $\rightarrow$ could be enhanced by π^2 due to Glauber phase



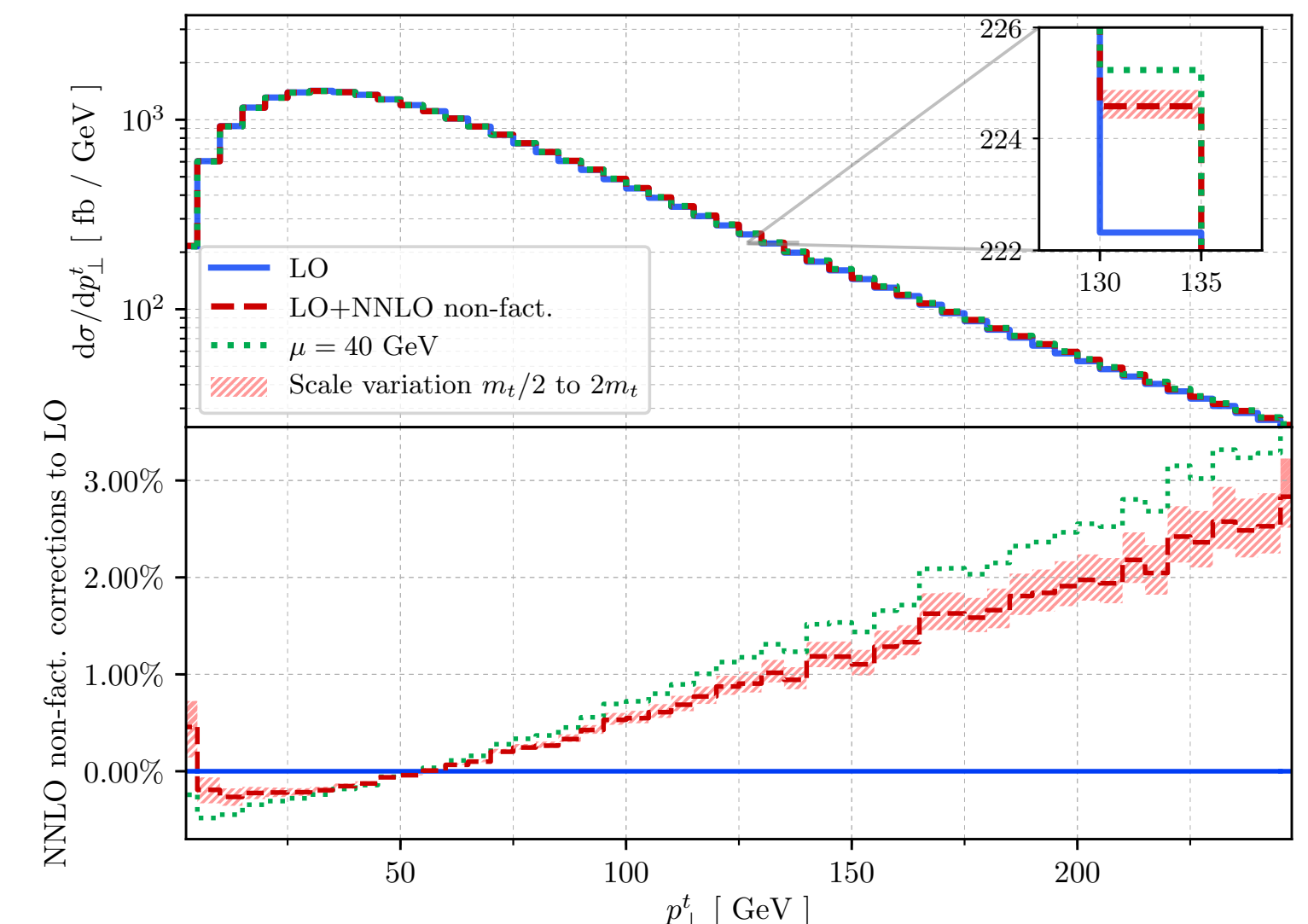
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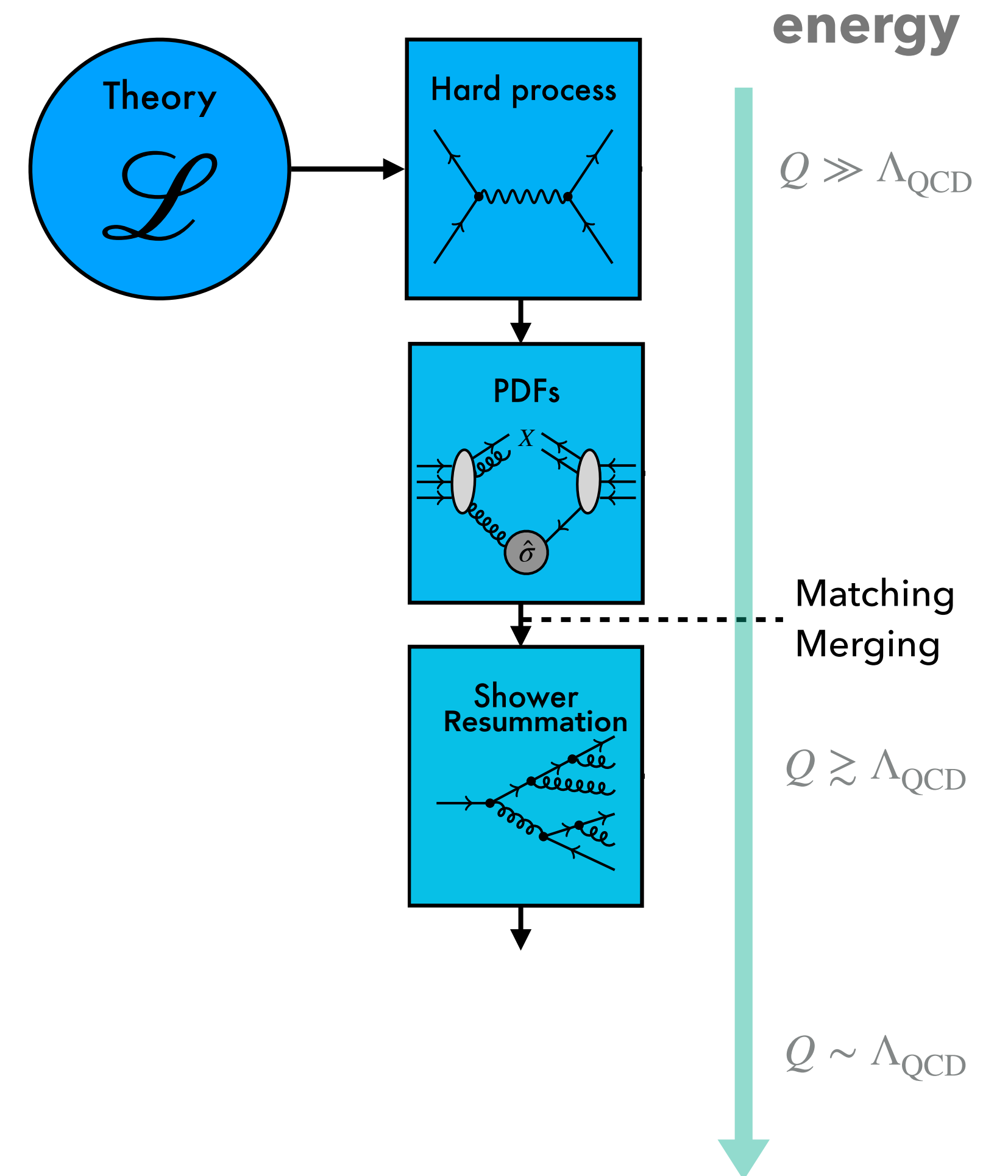


- Non-factorisable corrections are $0.22_{-0.04}^{+0.05}\%$ LO for $\mu_R = m_t$
- $\mu_R = 40\text{GeV}$ (typical transfer momentum scale of top):
 $\sigma_{\text{nf}}^{\text{NNLO}} = 0.35\% \cdot \sigma^{\text{LO}}$
- In comparison, NNLO factorisable corr. are around 0.7% NLO

conceptually and practically interesting $\rightarrow$ other processes can feature similar behaviour



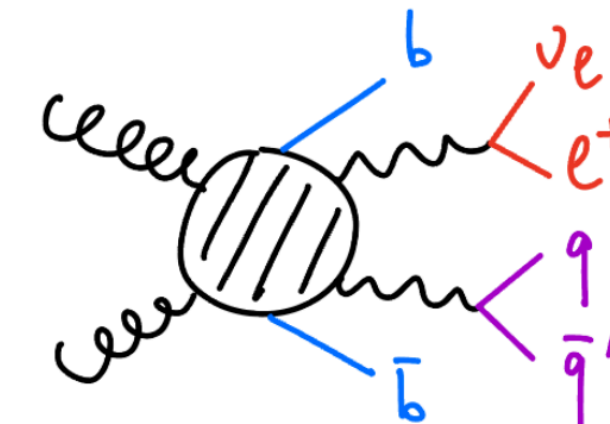
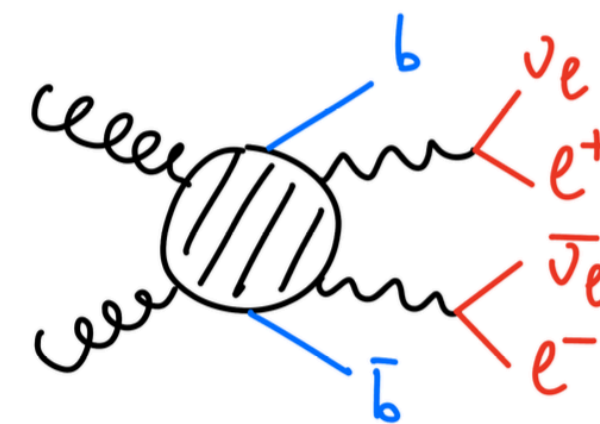
QUICK DETOUR BEYOND FIXED-ORDER PREDICTIONS



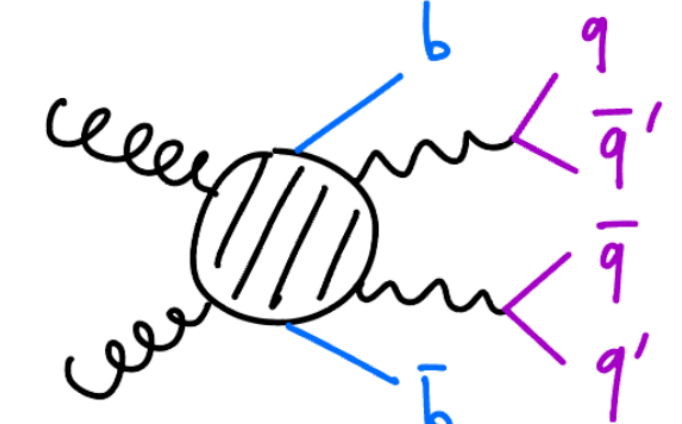
PRECISION & REALISM $pp \rightarrow b\bar{b}4l$

- ▶ LHC: top quark is predominantly produced in pairs
- ▶ Measurements of inclusive and differential cross sections, asymmetries, **parameter extraction** (α_s, m_t, y_t)
- ▶ Experimental signature: two bottom jets and the decay modes of W bosons

cleanest signature
QCD corrections "easier",
smaller cross-section



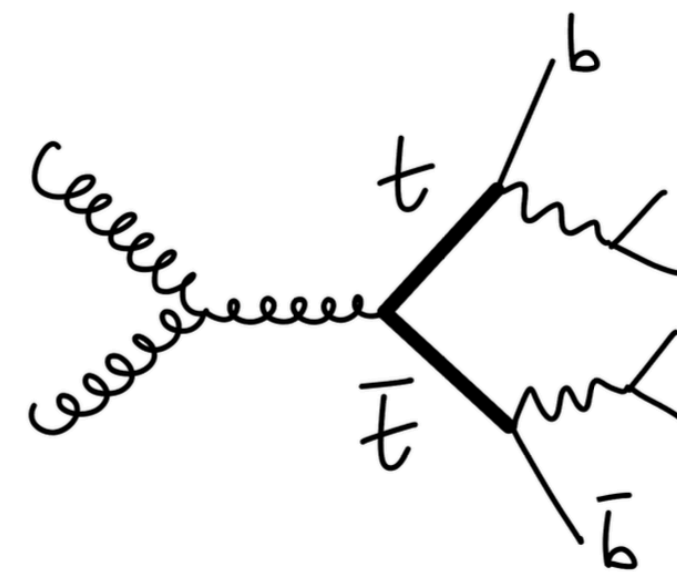
too many jets
Harder to model in pQCD
more abundant



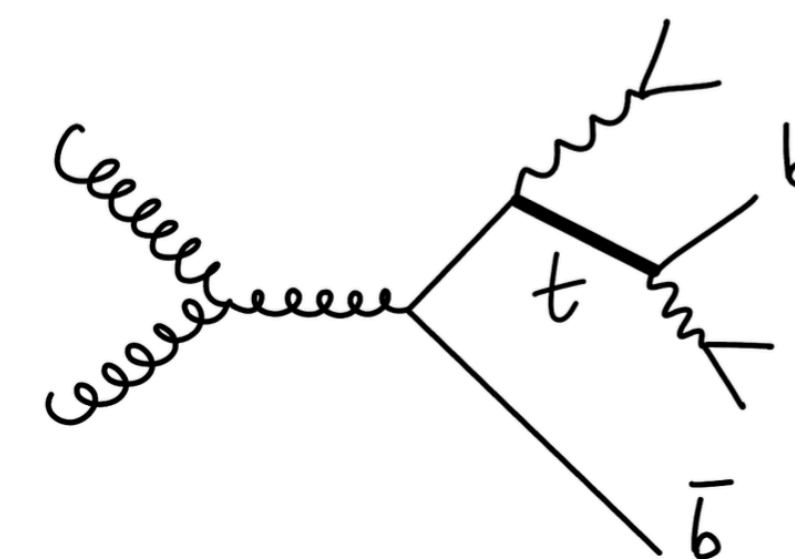
- ▶ **High number of partons** ($2 \rightarrow 6$ at Born level) : gigantic phase space to sample $pp \rightarrow b\bar{b}e^-\bar{\nu}_e\mu^+\nu_\mu(+1j,2j)$

- ▶ **Modelling of unstable particles** (different topologies)

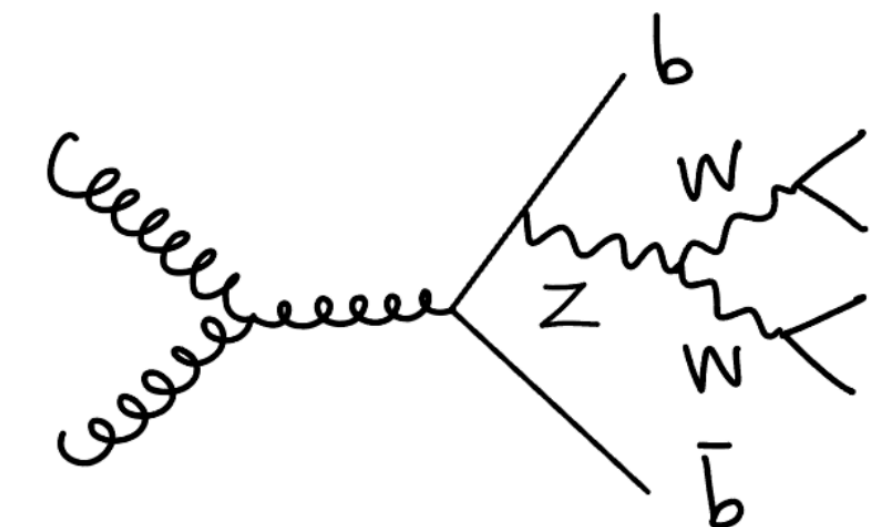
$$\mathcal{B}(t \rightarrow Wb) = \frac{\Gamma(t \rightarrow Wb)}{\Gamma(t \rightarrow Wq)} = \frac{|V_{tb}|^2}{|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2} \sim 0.99$$



Double top resonant



Non-resonant



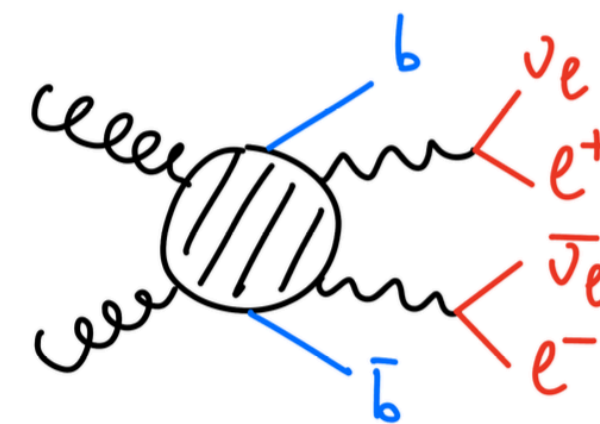
Single top resonant

- ▶ **Virtual corrections** extremely computationally costly or even unknown. 2loop, non-factorisable amplitudes out of reach
→ approximation? [\[Buonocore et al. 2507.11410, Ahmadova et al. 2605.09444\]](#)

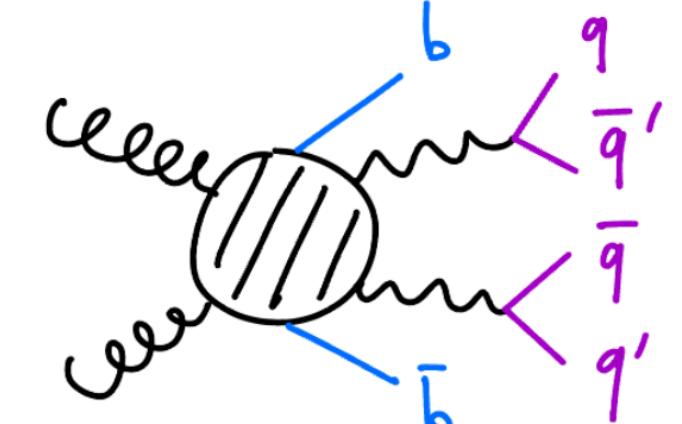
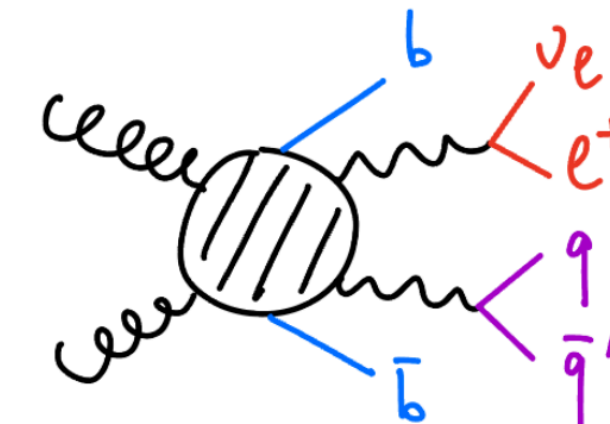
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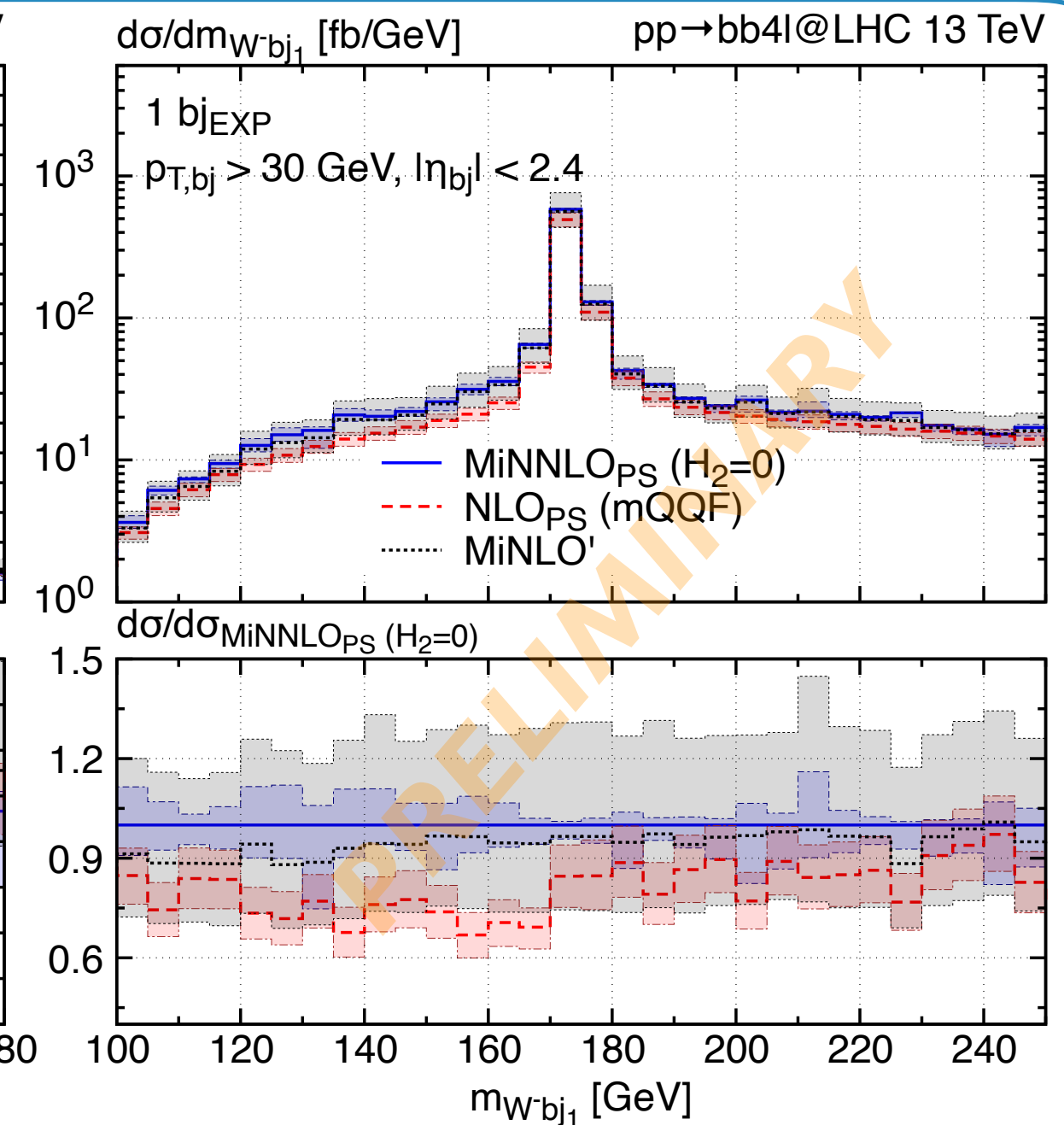
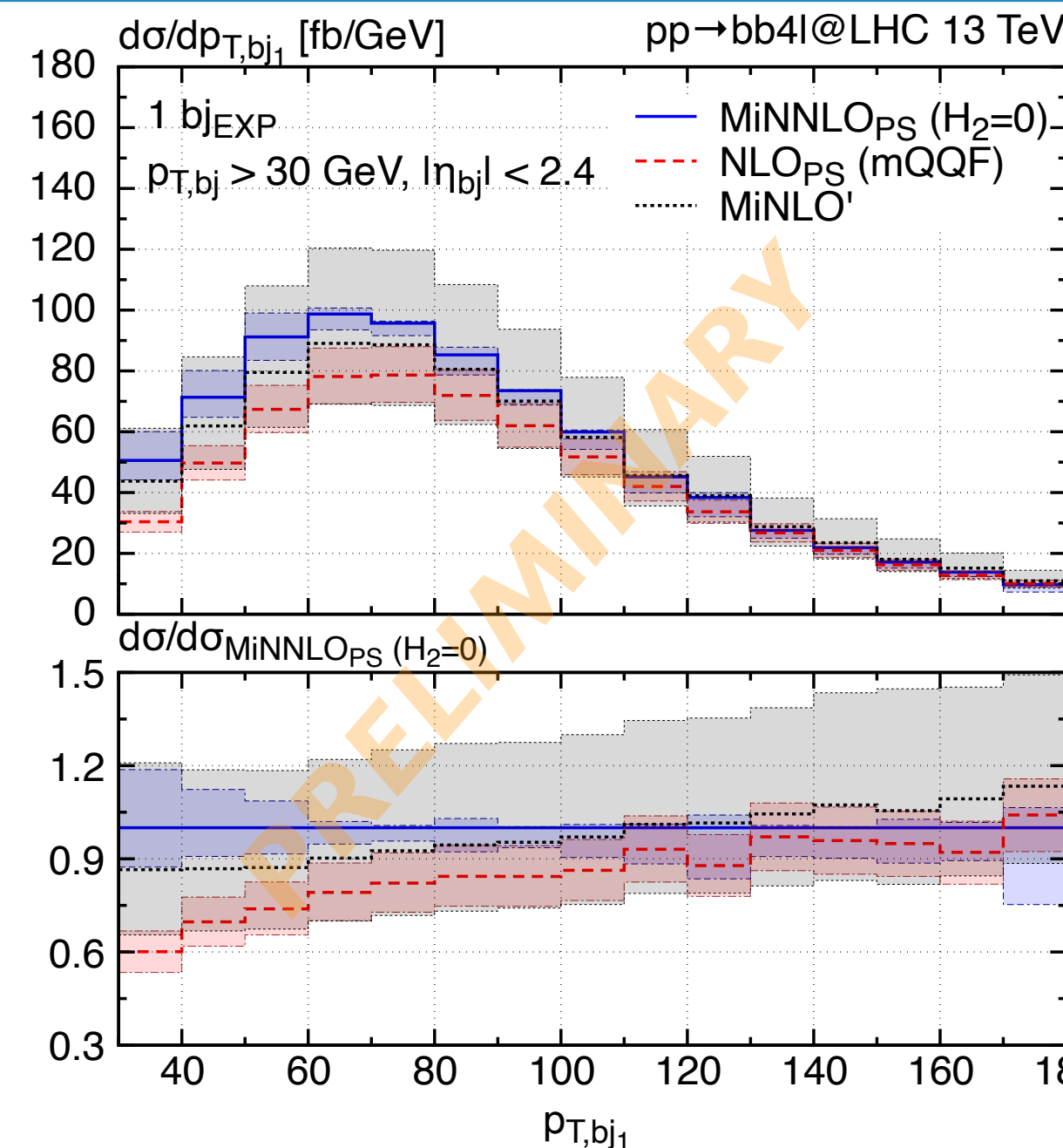
too many jets
Harder to model in pQCD
more abundant



[Biello, CSS, Wieseemann, Zanderighi, WIP]

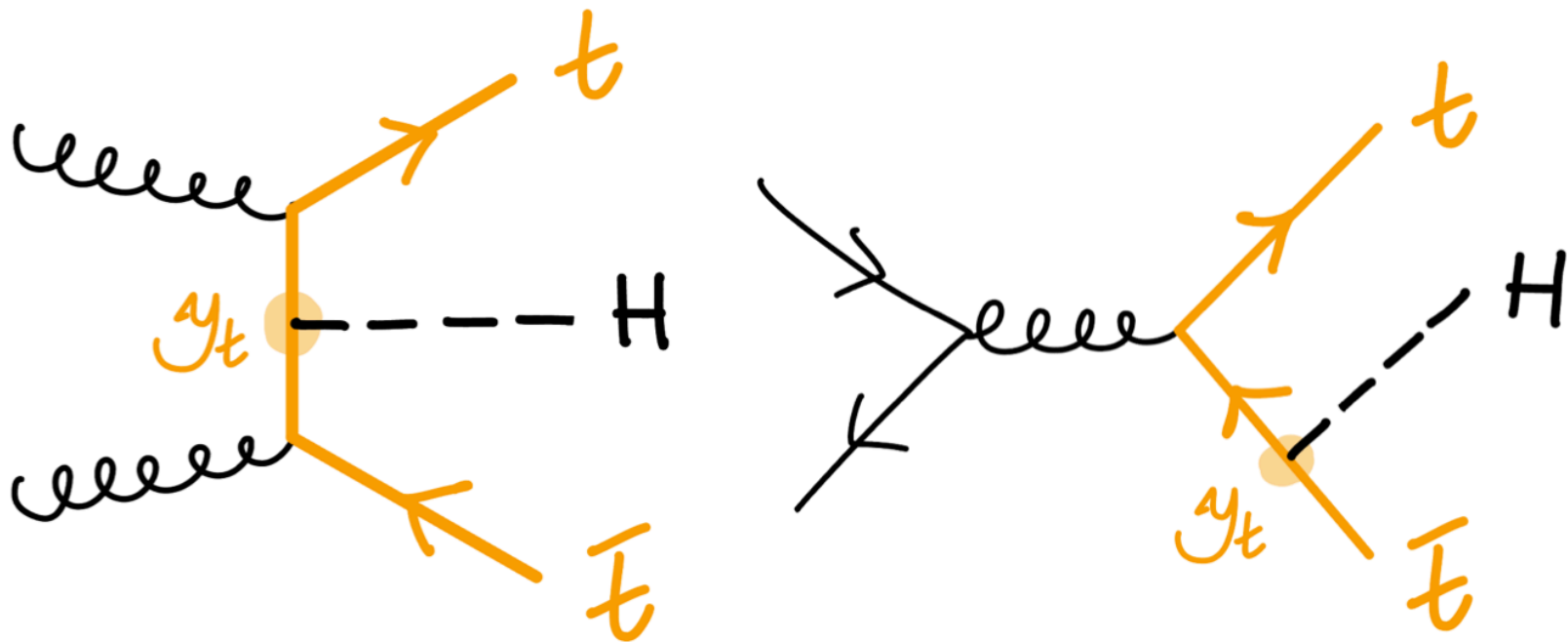
$pp \rightarrow b\bar{b}e^-\bar{\nu}_e\mu^+\nu_\mu$ (different leptonic flavour to reduce "Z pollution")

- ▶ At low $p_{\perp, W+W-b\bar{b}}$ MiNNLO reduces the NLO predictions (a spread of the peak is also evident)
- ▶ At high $p_{\perp, W+W-b\bar{b}}$ MiNNLO and MiNLO' coincide, both NLO accurate.
- ▶ MiNNLO turns to have up to $\sim 20\%$ effect wrt MiNLO' and NLO.



PRECISION&REALISM $pp \rightarrow t\bar{t}H$

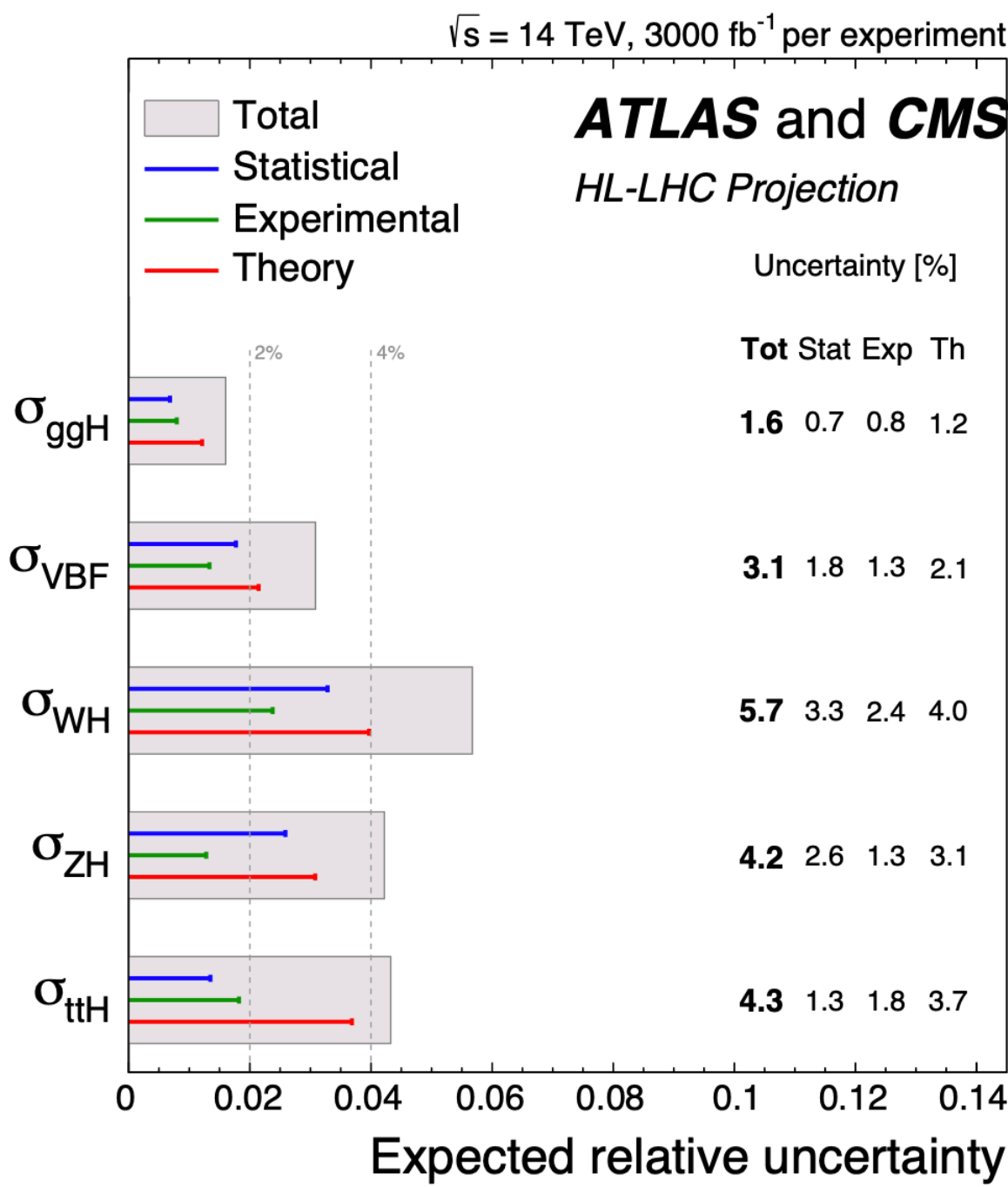
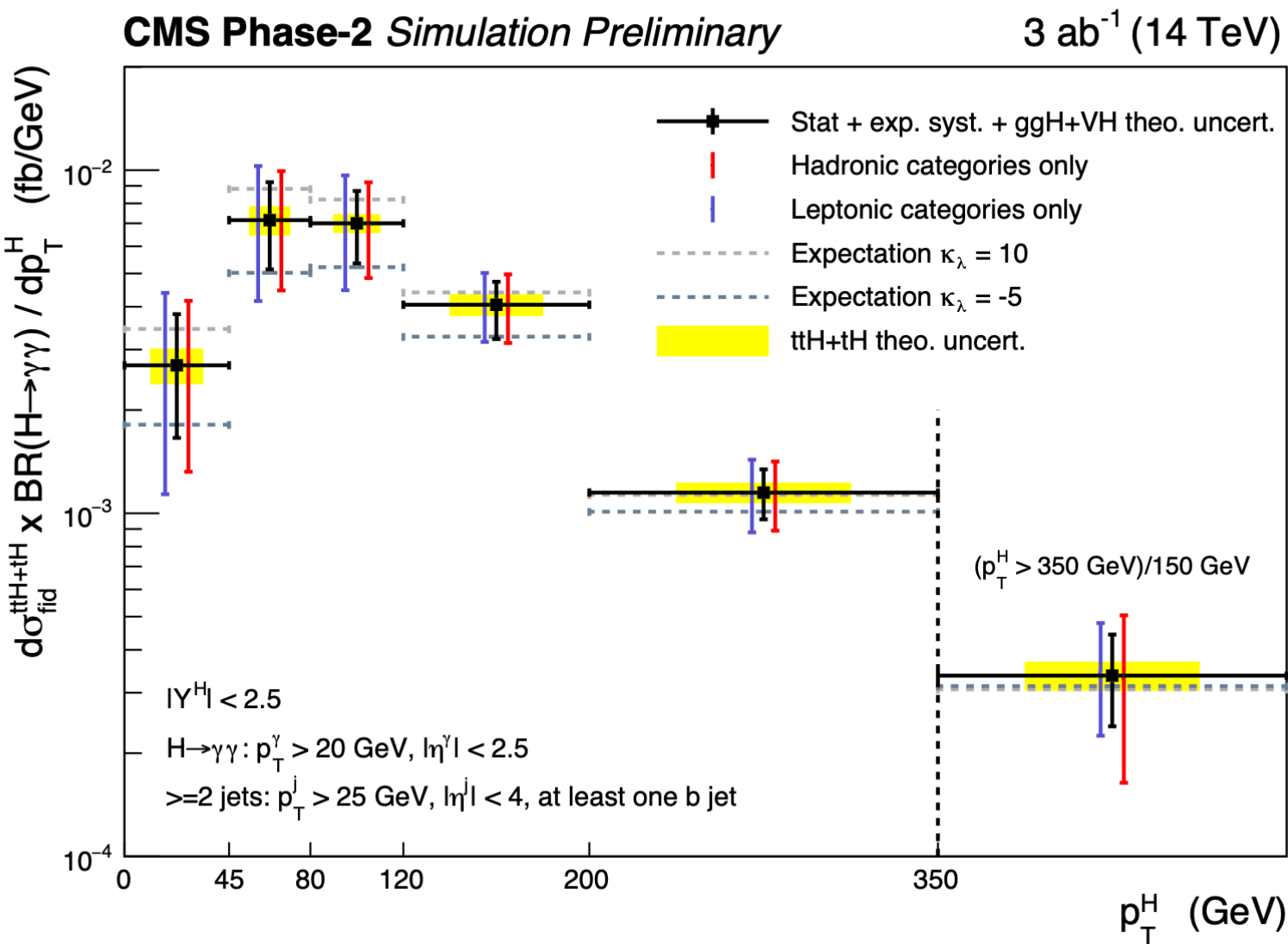
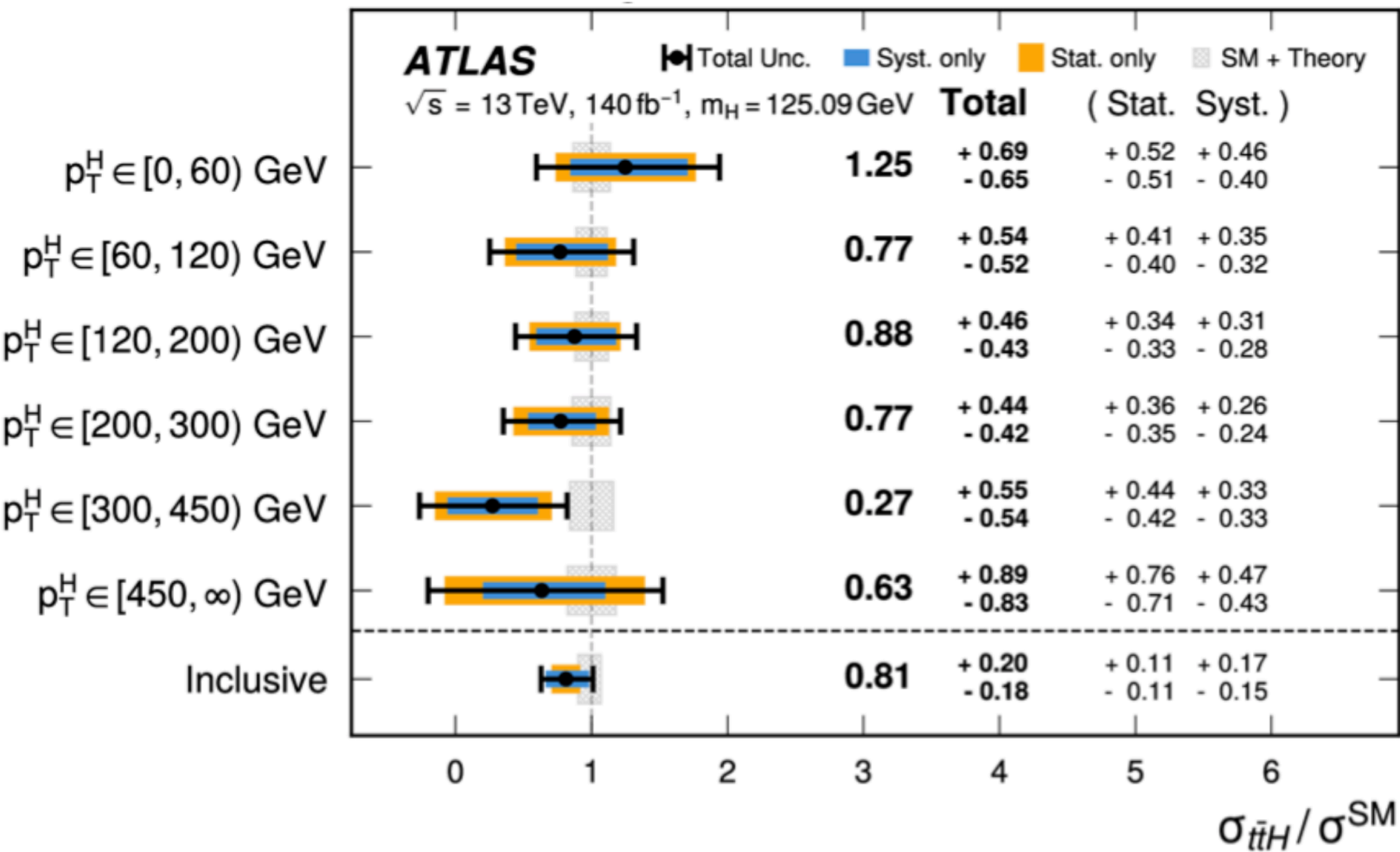
- allows for direct measurement of the top-quark Yukawa coupling
- important background to Higgs-pair searches
- ATLAS and CMS: $\mathcal{O}(20\%)$ accuracy for signal, expected $\mathcal{O}(2\%)$ @HL-LHC
- Current theoretical uncertainties are $\mathcal{O}(10\%)$ [LHC Higgs WG 1610.07922]



First differential ATLAS and CMS measurements [2407.10896, 2407.10904]

Present status [2510.23755]

Projections for the future [1902.00134]

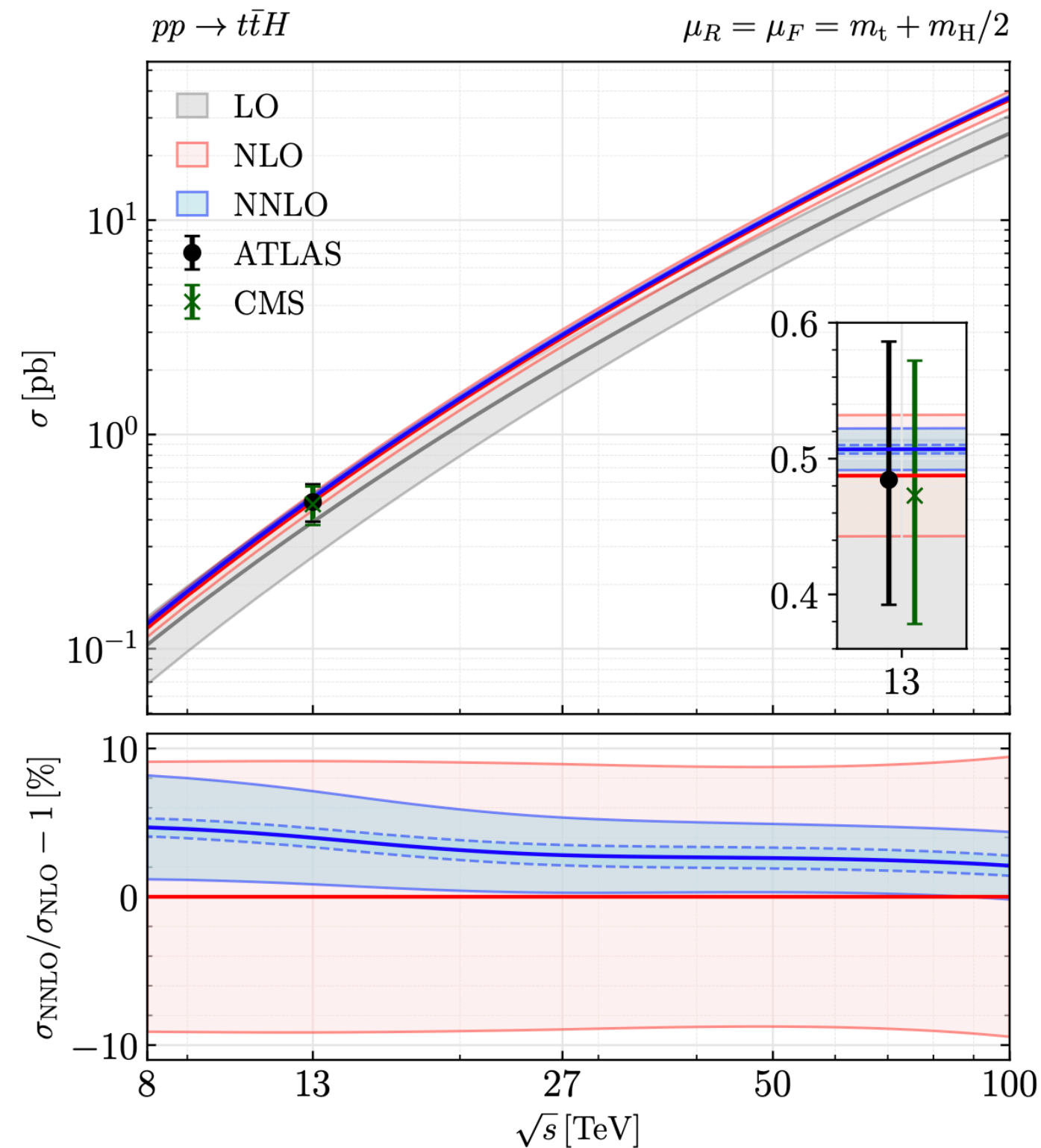


PRECISION&REALISM

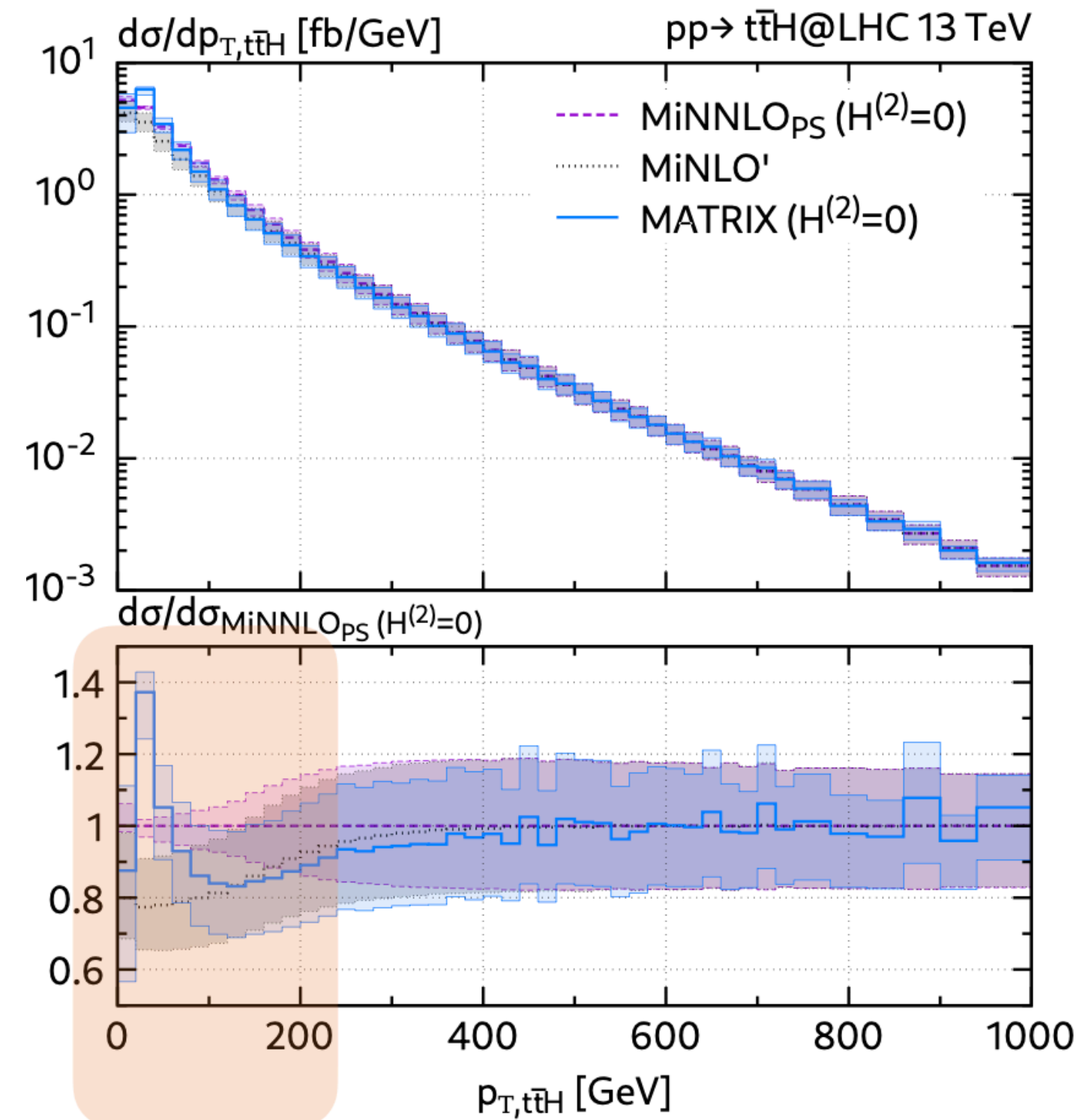
$$pp \rightarrow t\bar{t}H$$

[Biello, Savoini, *CSS*,
Wiesemann 2603.06143]

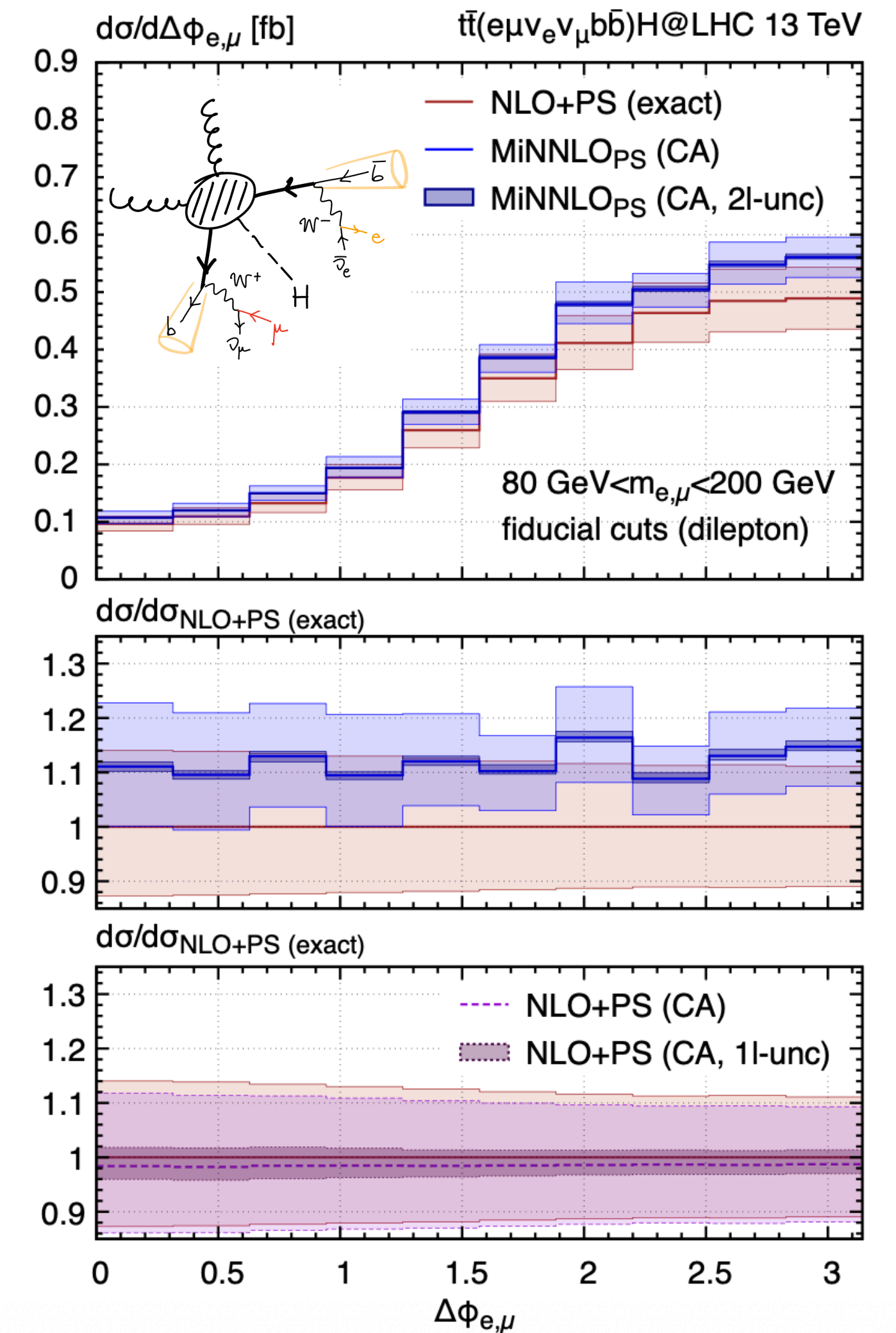
[(Catani), Devoto et al. 2210.07846, 2411.15340]



- NNLO large at xsection level, and larger at fiducial level:
 $\sim +10\%$ NLO @ $p_{\perp,H} = 200\text{GeV}$,
 uncertainty $\sim 1/3$ NLO



- Fixed-order predictions fails in the low- $p_{\perp}$ region $\rightarrow$ need for matching and resummation
- MiNNLOPS framework



- VV via combined (CA) approximation (soft Higgs and massification)
- Public code for exp. analysis

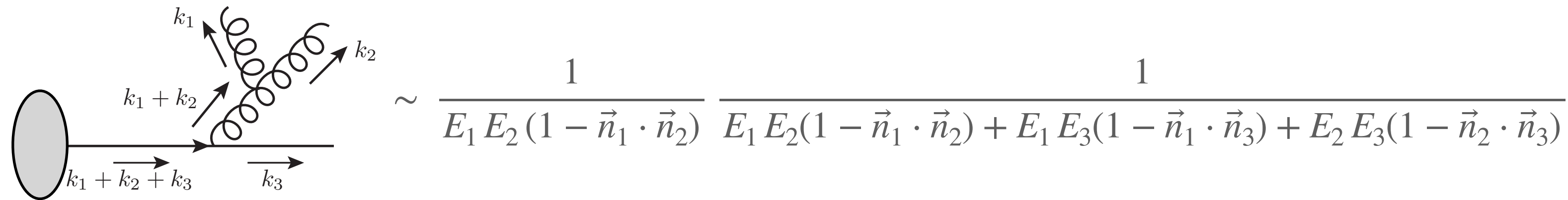
WRAPPING UP

- ❖ **Precision predictions are essential to fully exploit the LHC and future collider programmes**, requiring increasingly accurate higher-order calculations.
- ❖ **Infrared subtraction remains a key ingredient for fully differential computations**, and developing more **systematic, modular and efficient approaches** is essential to tackle **increasingly complex processes**.
- ❖ These advances already have a **direct impact on phenomenology**, enabling more accurate predictions and **revealing effects that were neglected in the past**, with important implications for precision Standard Model tests and searches for new physics.
- ❖ **Looking ahead**, *developing fully general, automated and community-ready methods will be crucial to match the experimental precision of the HL-LHC and future collider programmes, while bringing NNLO calculations to the level of maturity, flexibility and automation of NLO predictions.*

THANK YOU FOR LISTENING



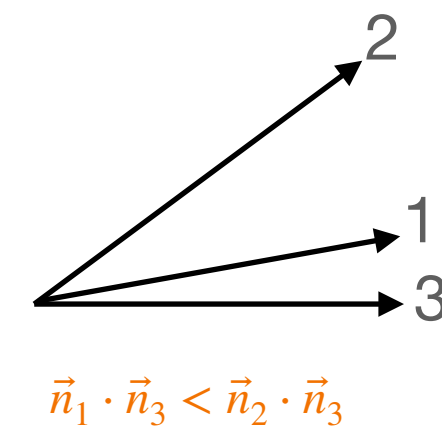
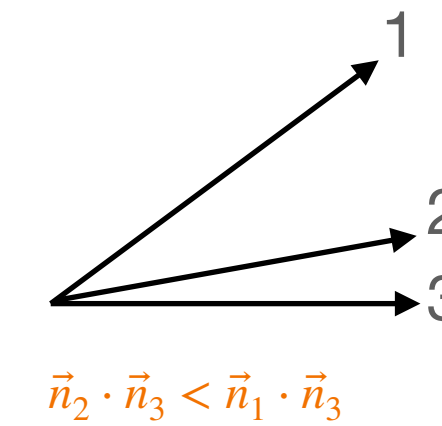
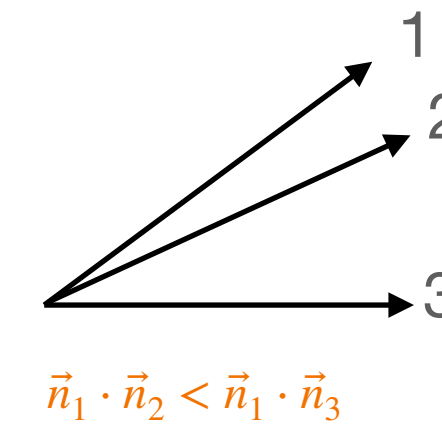
REAL CORRECTIONS@NNLO



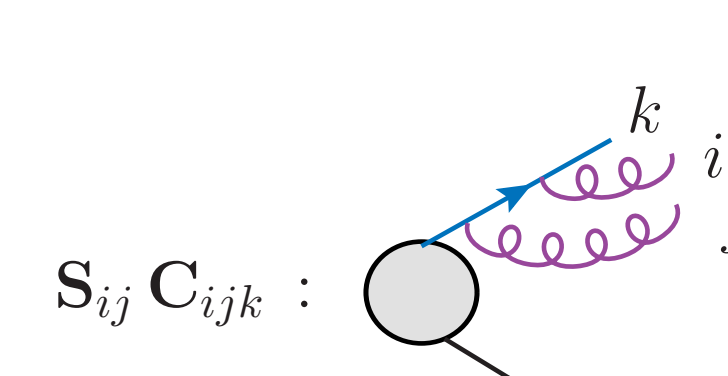
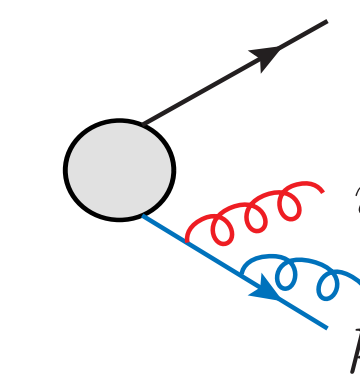
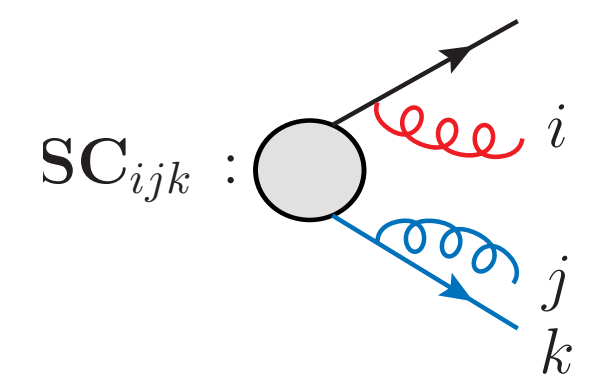
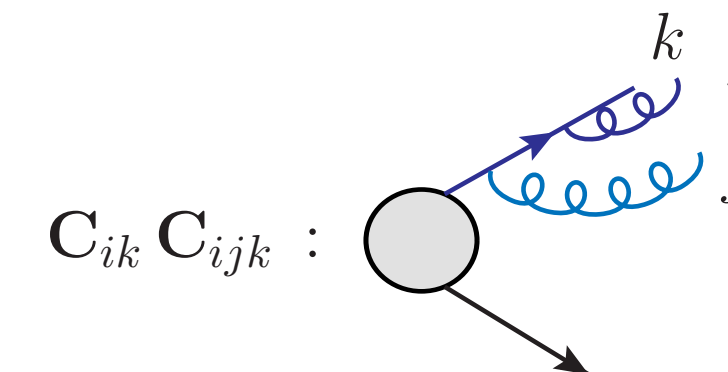
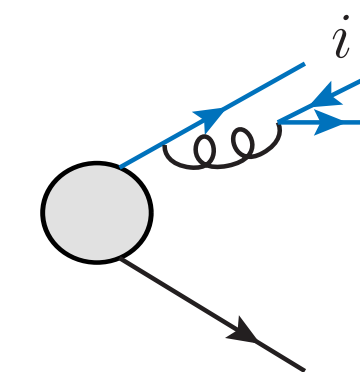
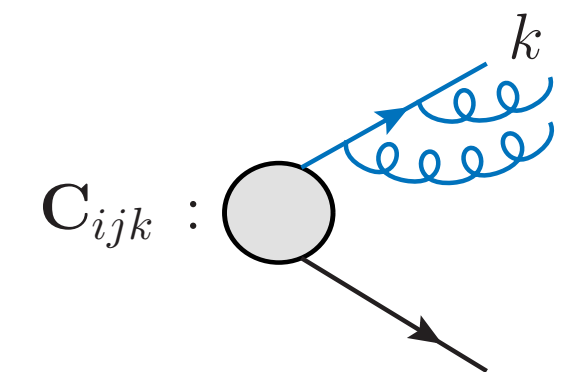
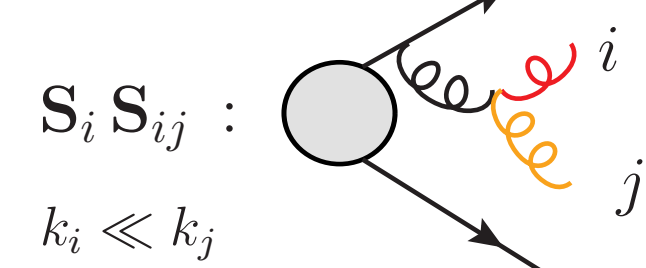
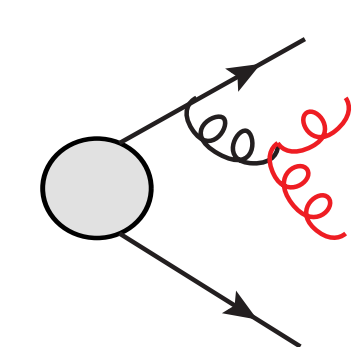
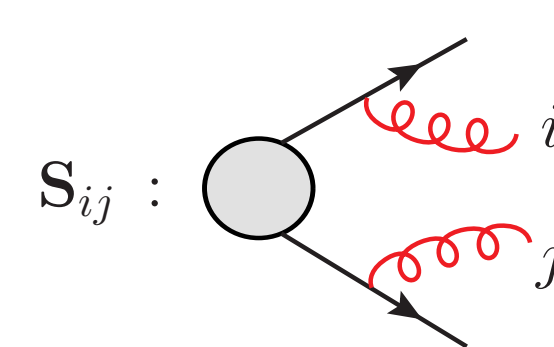
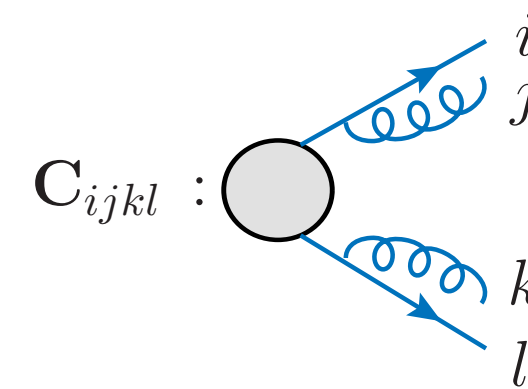
$$E_1 \rightarrow 0 \quad E_2 \rightarrow 0 \quad E_1, E_2 \rightarrow 0$$

$$\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$$

$$E_1 \ll E_2, \quad E_2 \ll E_1$$



1. Clear understanding of which singular configurations do actually contribute
2. Collect the relevant ingredients
3. Get to the point where the problem is well defined
4. Solve the phase space integrals of the relevant limits



MIXED CORRECTIONS

MAIN MESSAGES

- ▶ Mixed QCD×EW corrections to the Drell-Yan process are important to increase our control on SM parameters and search for new physics

PRECISION PROGRAM

LHC as precision machine

► m_W measurements:

High-precision measurement of the W boson mass with the CMS experiment at the LHC

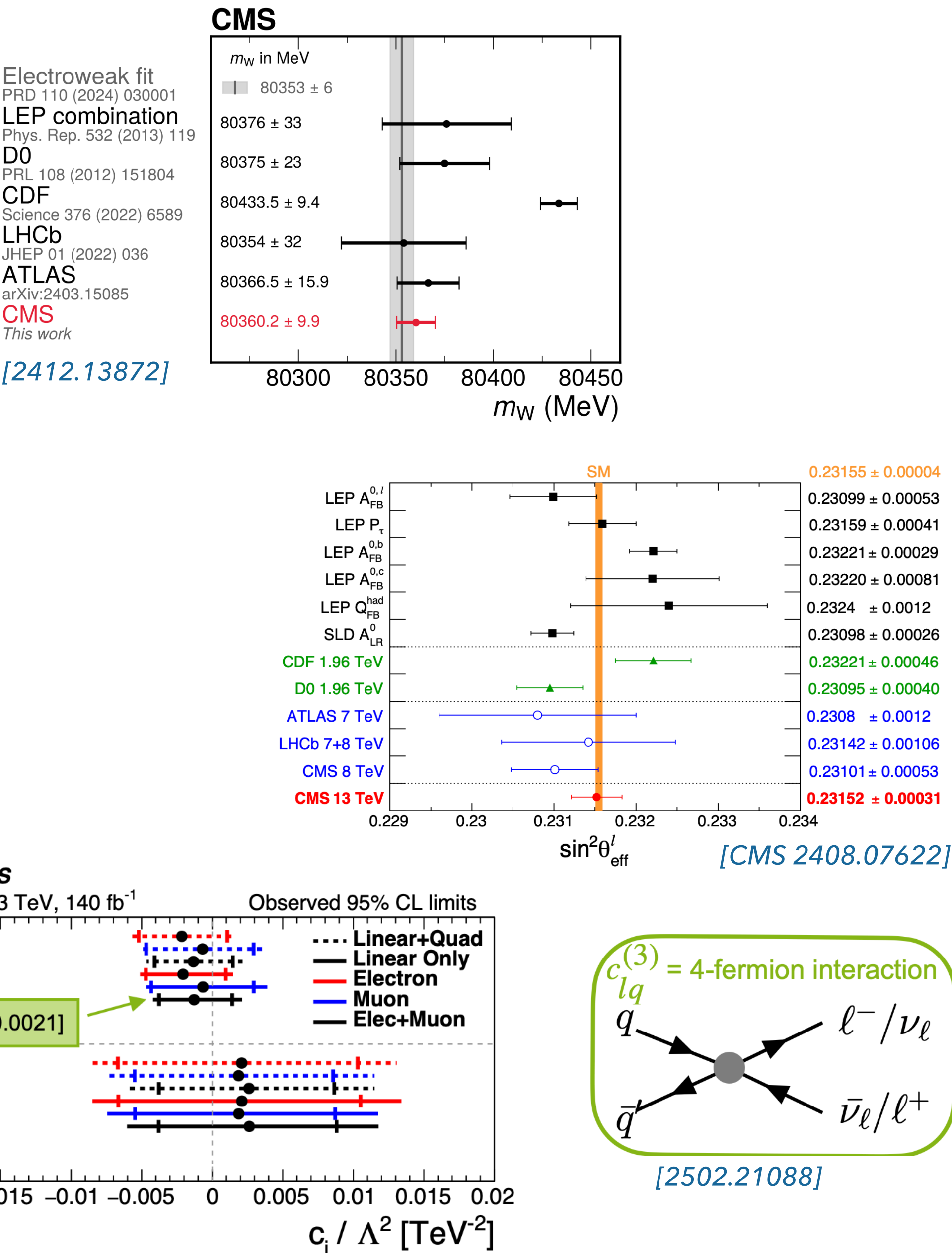
We report the first W boson mass measurement by the CMS Collaboration at the CERN LHC, based on a data sample collected in 2016 at the proton-proton collision energy of 13 TeV. The W boson mass is measured using a large sample of $W \rightarrow \mu\nu$ events

► $\sin^2 \theta_W$ measurements:

Measurement of the Drell–Yan forward-backward asymmetry and of the effective leptonic weak mixing angle in proton-proton collisions at $\sqrt{s} = 13$ TeV

► SMEFT coefficient constraints:

Measurement of double-differential charged-current Drell–Yan cross-sections at high transverse masses in pp collisions at $\sqrt{s} = 13$ TeV with the ATLAS detector

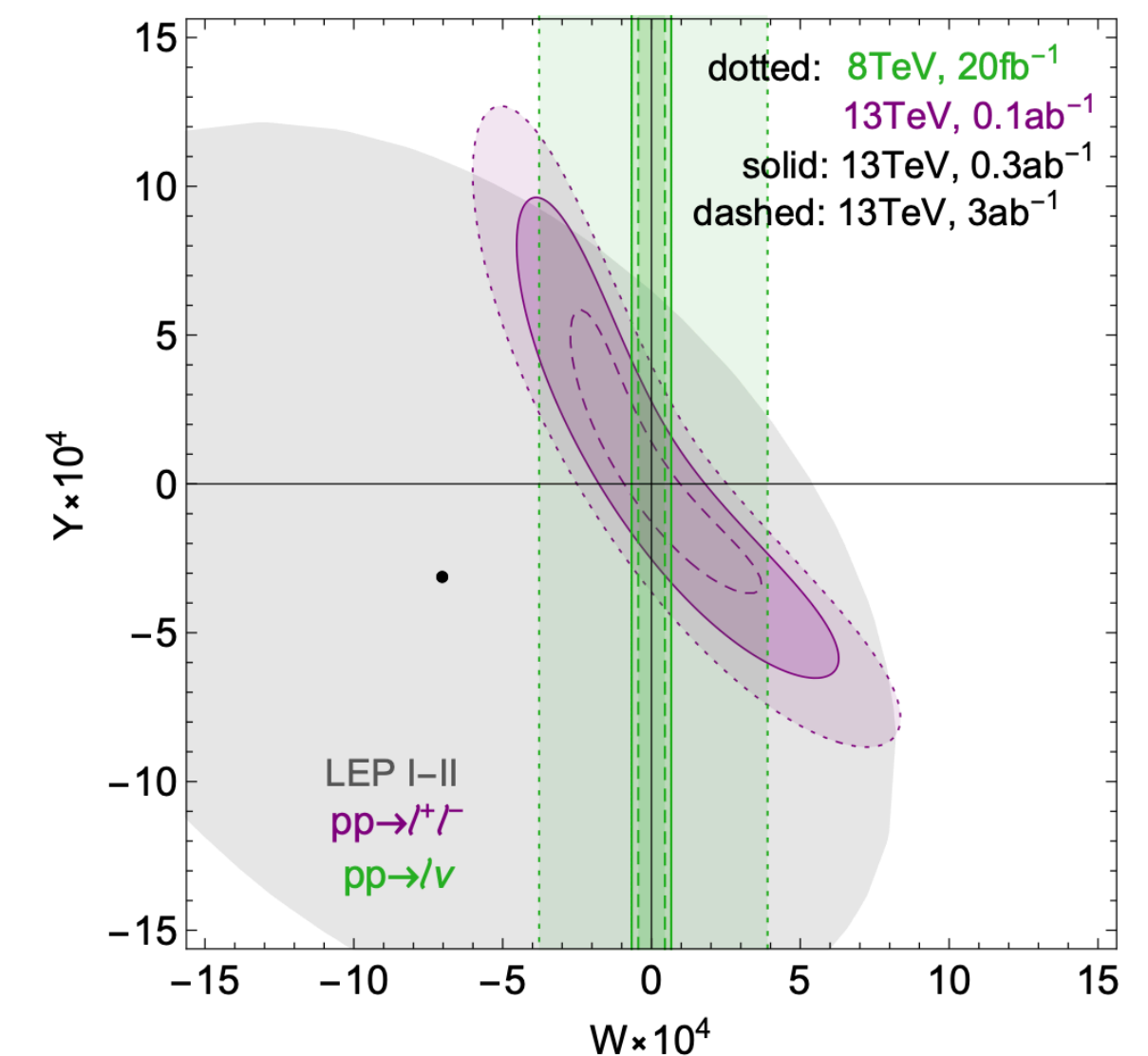
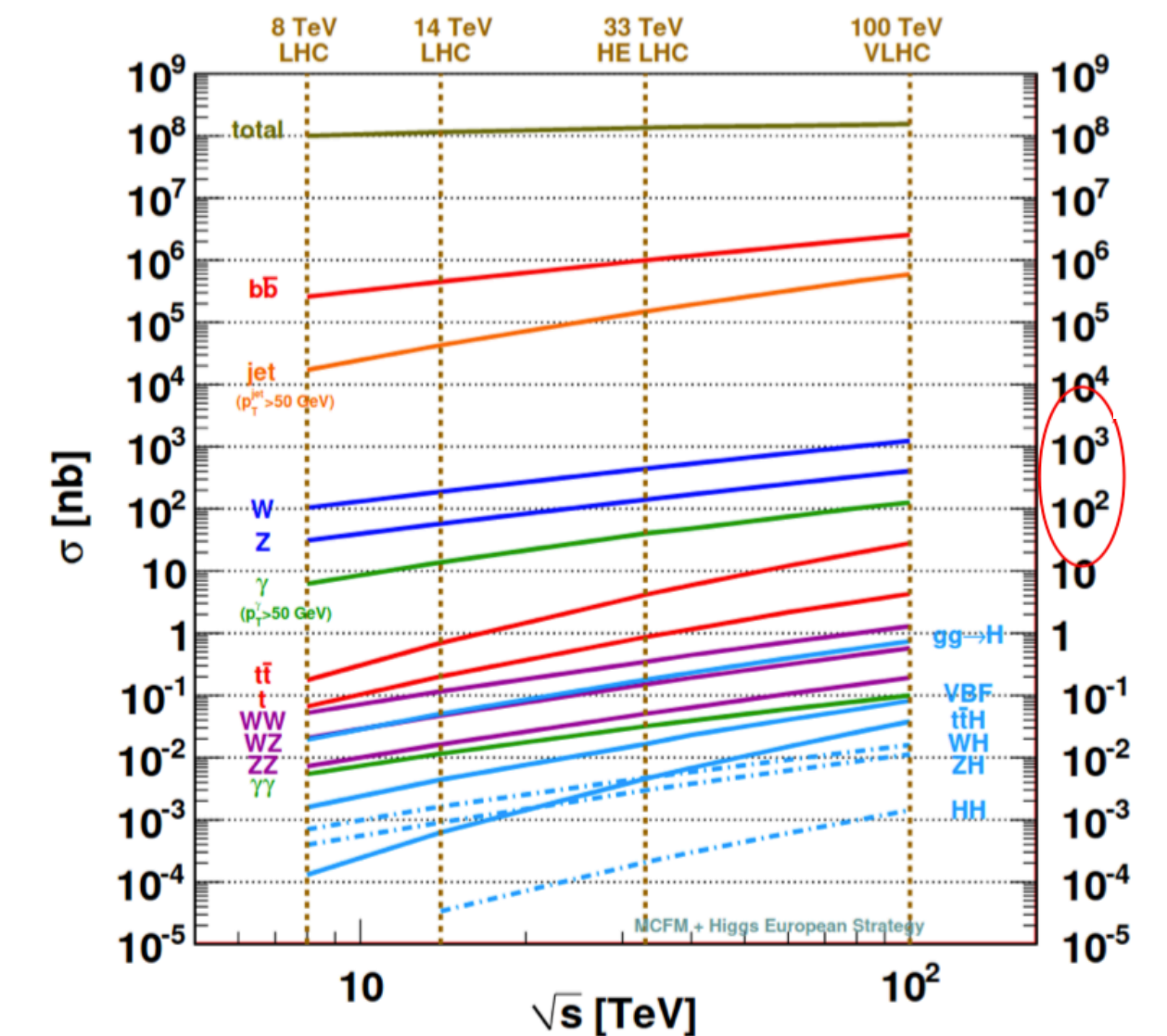


WHY DRELL-YAN PROCESS?

Drell-Yan process is one of the “standard candles” at the LHC:

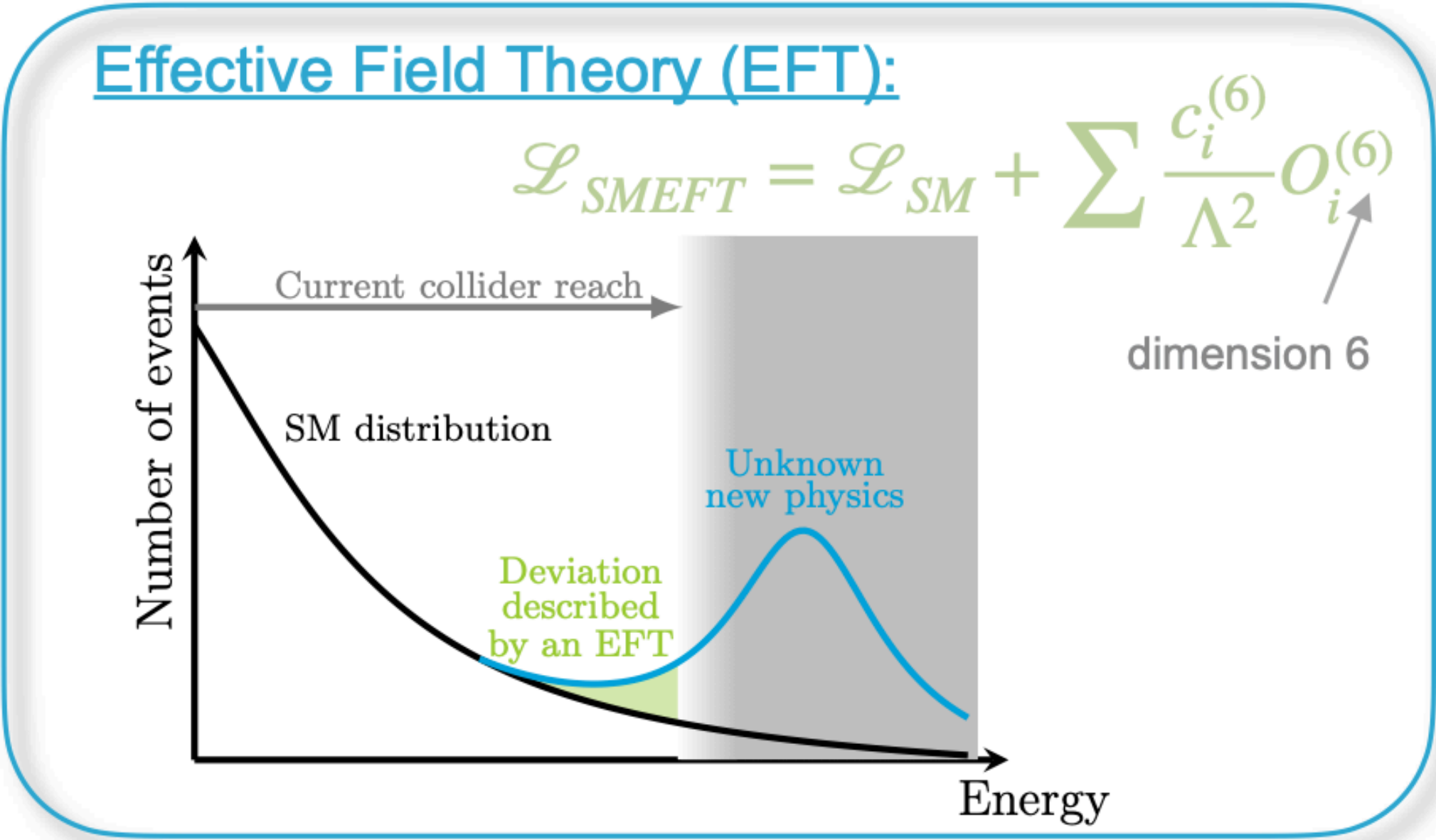
- ▶ **Clear experimental signature**, extremely well-controlled:
 - ▶ $Z \rightarrow \ell^+ \ell^-$: detection of a pair of charged leptons
 - ▶ $W \rightarrow \ell \nu$: one charged lepton + missing
- ▶ **Detector calibration** (e.g. detector response to lepton energy in NCDY events)
- ▶ **Large production rate**: relevant background for new physics searches also at future colliders
- ▶ Information about **PDFs**: separation of valence quarks through charge asymmetry
- ▶ Precise determination of the **EW SM parameters**: CCDY at resonance (m_W)
- ▶ Constrain **heavy NP** in a model-independent way using **SMEFT**

accurate theoretical predictions are necessary!

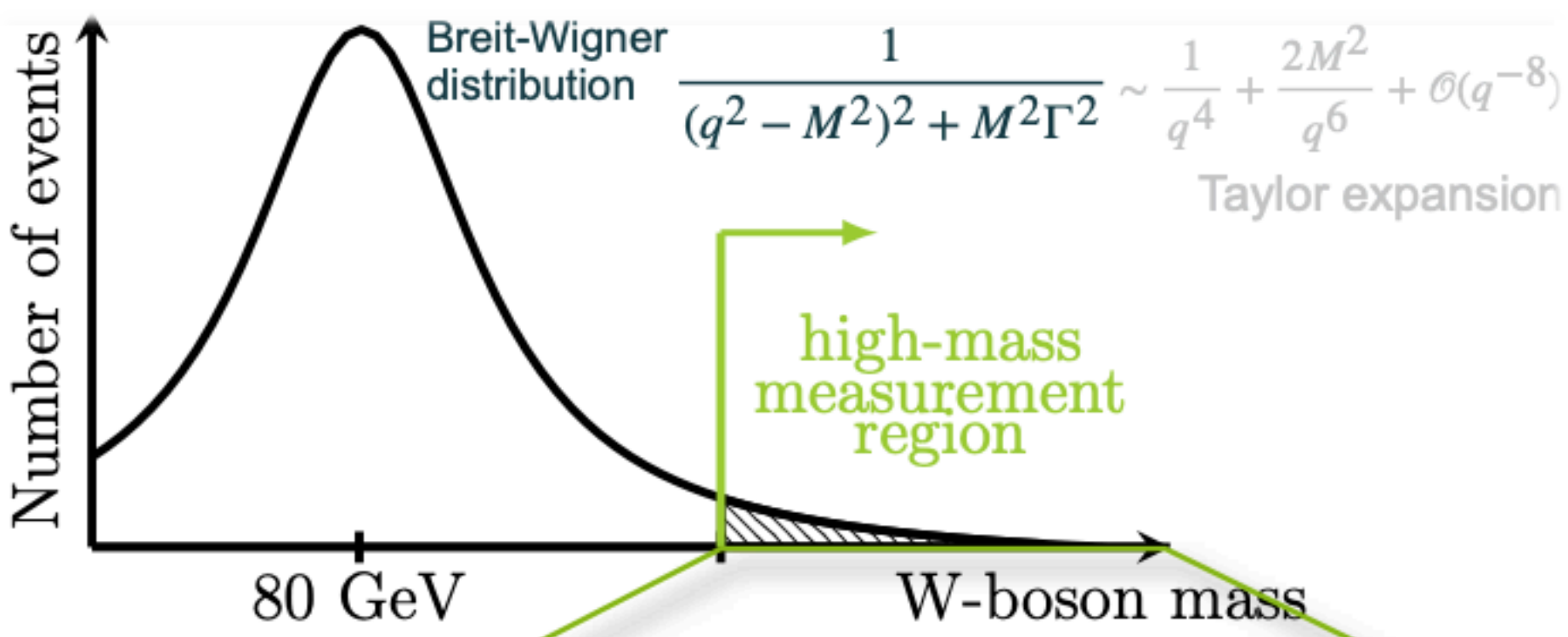


INCREASING EXP. INTEREST IN HIGH-ENERGY REGIONS

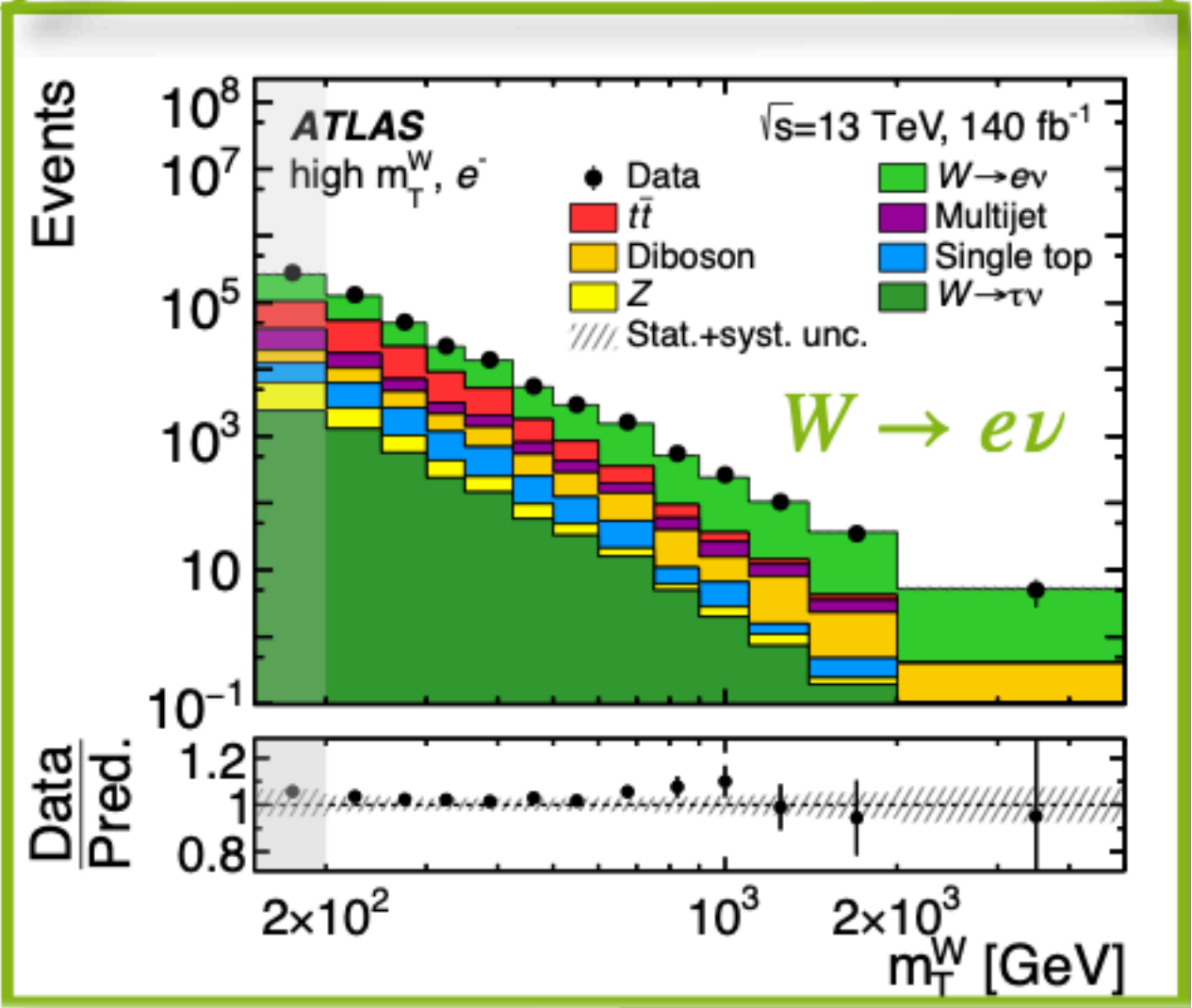
New (non-resonant) physics at high mass?



[Johanna Kraus | DIS 2025]



$m_T^W =$
200 GeV
- 5 TeV



[ATLAS 2502.21088]

INCREASING TH. INTEREST IN HIGH-ENERGY REGIONS: MIXED CORRECTIONS AS TRAIT D'UNION*

$$d\hat{\sigma}_{ij} = d\hat{\sigma}_{ij,LO} \left(1 + \alpha_s \Delta_{ij,NLO}^{QCD} + \alpha_{ew} \Delta_{ij,NLO}^{EW} + \alpha_s^2 \Delta_{ij,NNLO}^{QCD} + \boxed{\alpha_s \alpha_{ew} \Delta_{ij,NNLO}^{QCD \otimes EW}} + \alpha_s^3 \Delta_{ij,N3LO}^{QCD} + \dots \right)$$

Couplings: $\alpha_s \sim 0.1$, $\alpha_{ew} \sim 0.01$

Mixed QCD×EW corrections:
 $\sim \mathcal{O}(\alpha_s \alpha_{ew}) \sim 0.1 \%$

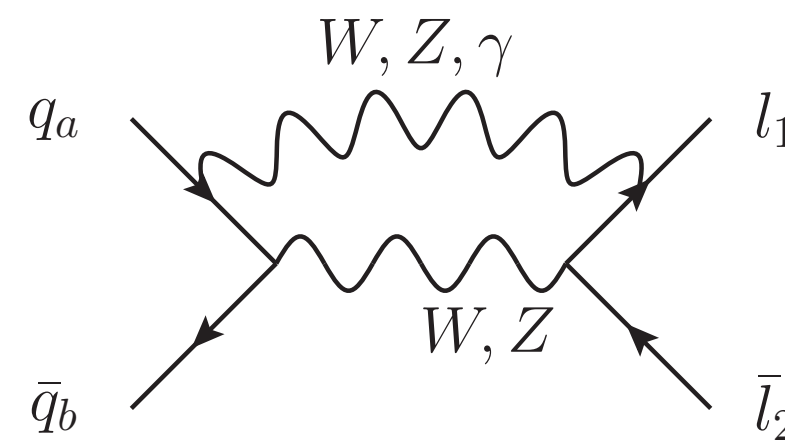
More accurate expectation:
(20% QCD) × (10% EW)

Potential **2%** correction in the
 high mass region

At high energy, this naive power counting is spoiled by **Sudakov logarithms**:

in EW theory $m_{W/Z}$ provides a physical cut-off: real Z, W can be detected as distinguishable particles.

Large logs from virtual corrections are of physical significance and **spoil the naive power counting** [Kuhn, Penin, Smirnov '00][Ciafaloni, Ciafaloni, Comelli '01][Denner, Pozzorini '01]



$$\sqrt{s} = 1 \text{ TeV} \gg m_W$$

$$\frac{\alpha_{ew}}{4\pi \sin^2 \theta_W} \log^2 \left(\frac{s}{m_W^2} \right) \sim 6 \%$$

$$\frac{\alpha_{ew}}{4\pi \sin^2 \theta_W} \log \left(\frac{s}{m_W^2} \right) \sim 1 \%$$

*Long but still non-exhaustive list of references on recent calculations in the backup

MAIN MESSAGES

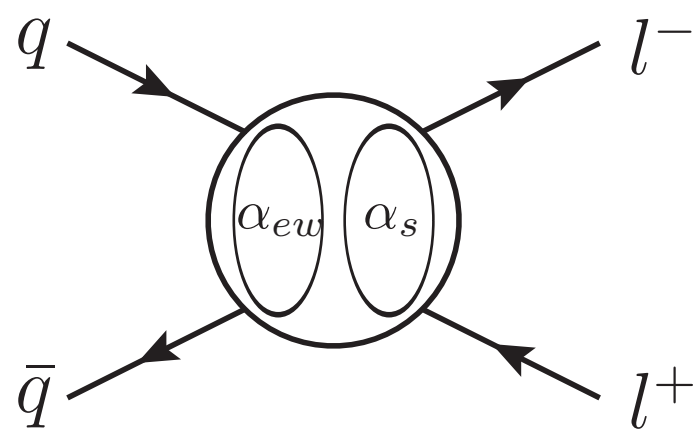
- ▶ Mixed QCD×EW corrections to the Drell-Yan process are important to increase our control on SM parameters and search for new physics
- ▶ Theoretical calculations are reaching the required level of sophistication to provide prediction at sub-percent accuracy. Crucial role played by developments in QCD beyond NLO → importance of cross-checks among different methods

CHALLENGES OF THE CALCULATION

Fixed-order calculation of mixed corrections involves two main technical challenges:

Amplitudes:

Two-loop integrals involving various masses



Two independent approaches:

Fully analytic calculation of 2loop integrals and helicity amplitudes [**NCDY**: [Heller et al. 1907.00491](#), [Heller et al. 2012.05918](#)]

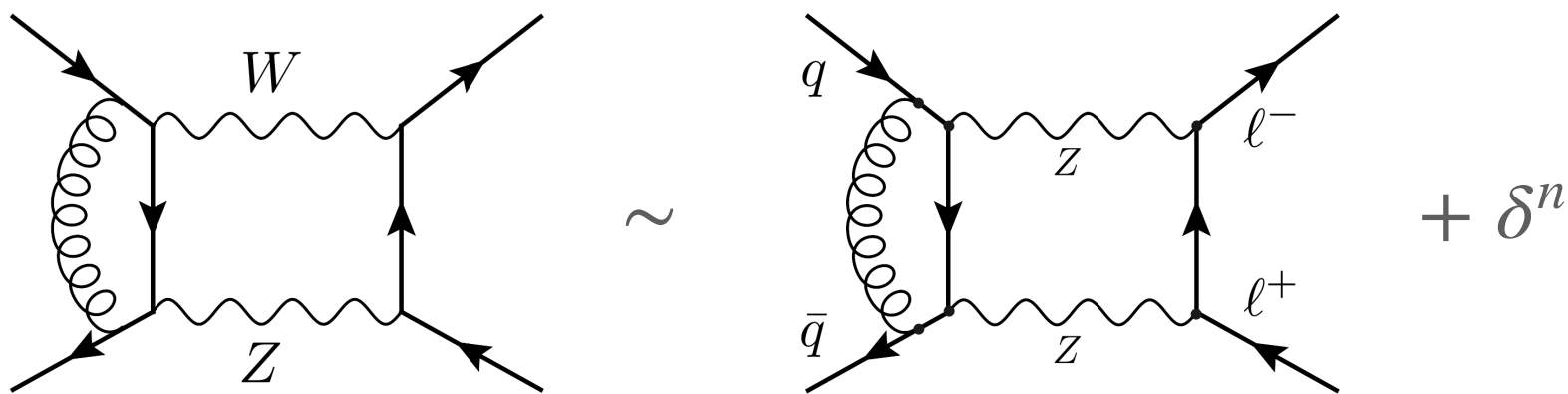
- Result in terms of GPLs
- Fast to evaluate ($\sim 0.8s$ /psp on standard laptop)
- Need some extra care with thresholds

Master integrals computed via **generalised series expansion** [**NCDY**: [Armadillo et al. 2201.01754](#), **CCDY**: [Armadillo et al. 2405.00612](#)]

- Analytical/numerical boundary conditions+evolution via DE
- Grid implementation for actual Monte Carlo runs
- Results already available for both NC and CC

Extension to CCDY*:

- Integrals with 1 mass $\rightarrow$ known from NCDY
- Integrals with 2 masses $\left\{ \begin{array}{l} \bullet m_1 = m_2 \rightarrow \text{known from NCDY} \\ \bullet m_1 \neq m_2 \rightarrow \text{expansion in } \delta = 1 - m_W/m_Z \\ (m_W = m_Z \text{ boundary integrals already known}) \end{array} \right.$



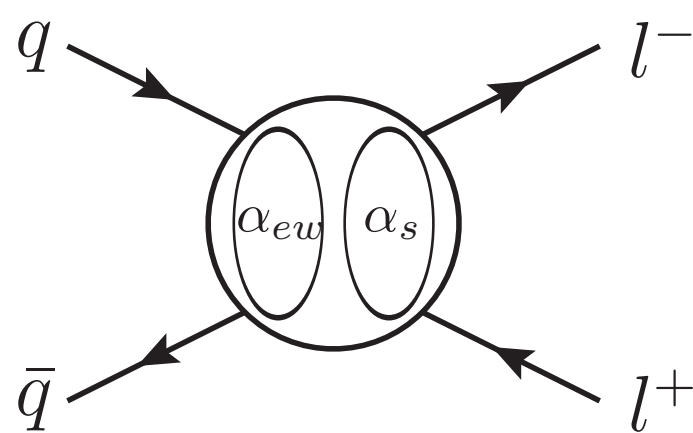
*two-loop integral families in the backup

CHALLENGES OF THE CALCULATION

Fixed-order calculation of mixed corrections involves two main technical challenges:

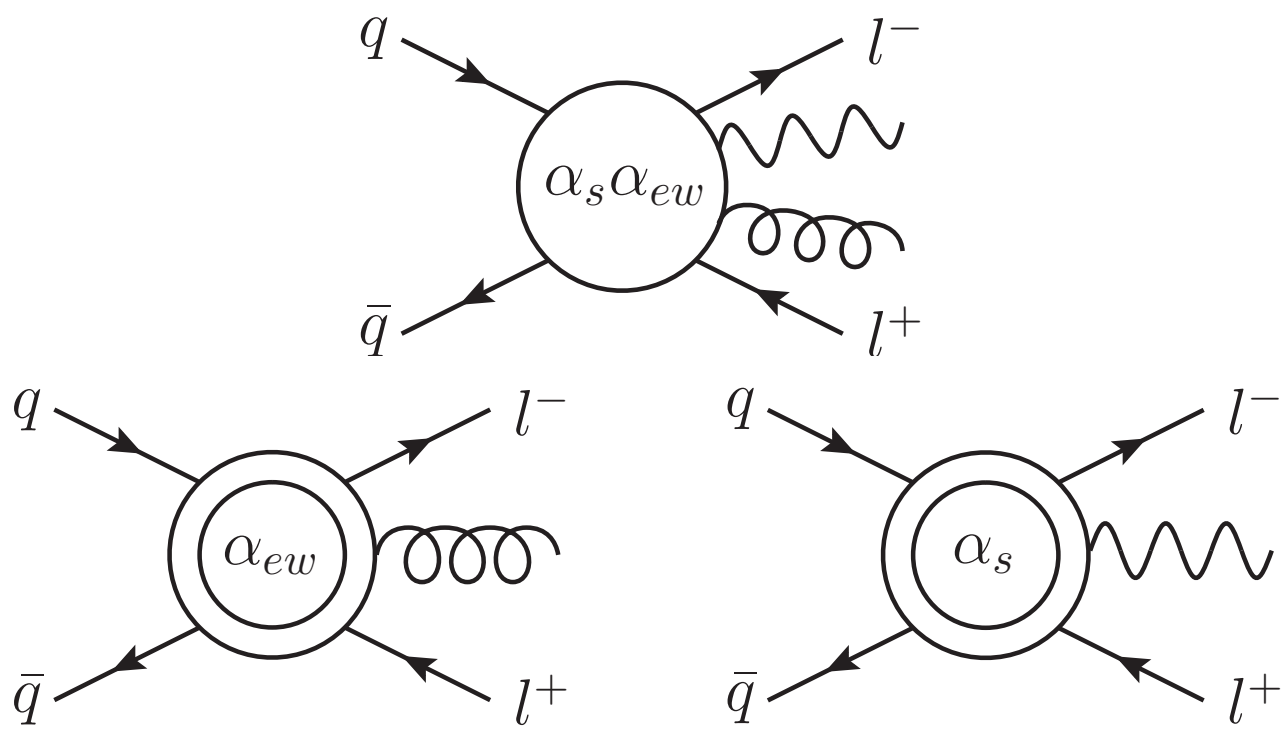
Amplitudes:

Two-loop integrals involving various masses



Real radiation:

Extraction of **soft/collinear** singularities
→ fully differential predictions
→ need to be suitable for phenomenology



Two independent calculations for NCDY:

[Luca Buonocore | CMS precision EWK workshop]

	Bonciani, Buonocore, Grazzini, Kallweit, Rana, Tramontano, Vicini '21	Buccioni, Caola, Chawdhry, Devoto, Heller, von Manteuffel, Melnikov, Röntsch, CSS '22
2loop amplitude	Semi-analytic: MIs via series expansion	Fully analytic
Real radiation	qT slicing (implemented in MATRIX): - Global cancellation of divergences: large systematics - Net reparation of NLO and NNLO singularities: recycling of NLO results & simpler NNLO counterterms massive leptons (mass as IR regulator)	Nested soft-collinear subtraction: Fully differential cancellation of divergences - More intricate structure of the subtraction counterterms @NNLO massless leptons (photon recombination)

A special thank to Luca and Federico for the intense cross-checking campaign

CHALLENGES OF THE CALCULATION

[Federico Buccioni | SM@LHC 2026]

Two independent calculations for NCDY:

High-invariant mass region

$m_{\ell\ell} > 200 \text{ GeV}, p_{T,\ell^\pm} > 30 \text{ GeV}, \sqrt{p_{T,\ell^+}p_{T,\ell^-}} > 35 \text{ GeV}, |y_{\ell^\pm}| < 2.5$

$\mu_R = \mu_F = m_{\ell\ell}/2 \quad \sqrt{s_{\text{had}}} = 13.6 \text{ TeV}$

relative impact of QCD×EW higher: ~ 0.8/1%

q_T slicing, dressed leptons [Buonocore et al 2412.16095]

σ [pb]	σ_{LO}	$\sigma^{(1,0)}$	$\sigma^{(0,1)}$	$\sigma^{(2,0)}$	$\sigma^{(1,1)}$
$q\bar{q}$	1561.52(5)	340.3(3)	-49.77(5)	44.6(4)	-15.7(7)
qg	—	0.0601(3)	—	-32.7(2)	2.15(7)
$q\gamma$	—	—	-0.30(2)	—	-0.231(9)
$g\gamma$	—	—	—	—	0.266(1)
gg	—	—	—	2.02(6)	—
$\gamma\gamma$	59.645(6)	—	3.174(9)	—	—

Comparison between two totally independent predictions (amplitudes, subtraction schemes)

- excellent agreement at LO, NLO QCD and EW and NNLO QCD
- excellent agreement for most channels in QCD×EW ($\gamma g, q\gamma, qg$)
- WIP to resolve mild tension in $q\bar{q}$ channel

Nested-soft collinear, dressed leptons [FB et al 2203.11237]

σ [fb]	$\sigma^{(0,0)}$	$\delta\sigma^{(1,0)}$	$\delta\sigma^{(0,1)}$	$\delta\sigma^{(2,0)}$	$\delta\sigma^{(1,1)}$
$q\bar{q}$	1561.42	340.31	-49.907	44.60	-14.78
$\gamma\gamma$	59.645		3.166		
qg		0.060		-32.66	2.07
$q\gamma$			-0.305		-0.227
$g\gamma$					0.2668
gg				1.934	

results updated wrt to publication (fixed implementation bug)

NLO QCD ~ +20%LO
→ $\delta^{\text{QCD}} \sim 8C_F \alpha_s / (2\pi) \sim 0.2$

NLO EW ~ -3%LO
→ $\delta^{\text{ew}} \sim \alpha_{ew} / \sin^2\theta_W \sim 0.03$

NNLO QCD ~ +0.8% LO
→ unexpected : $\alpha_s^2 \sim 0.014$
(strong qq and qg cancellation)

QCD×EW ~ -1% LO
→ unexpected : $\alpha_s \alpha_{ew} \sim 0.00089$

NESTED SOFT-COLLINEAR SUBTRACTION IN A NUTSHELL [CAOLA, MELNIKOV, RÖNTSCH 1702.01352]

Extension of **FKS subtraction** [Frixione, Kunst, Signer 9512328] to **NNLO** and inspired by **STRIPPER** [Czakon 1005.0274]

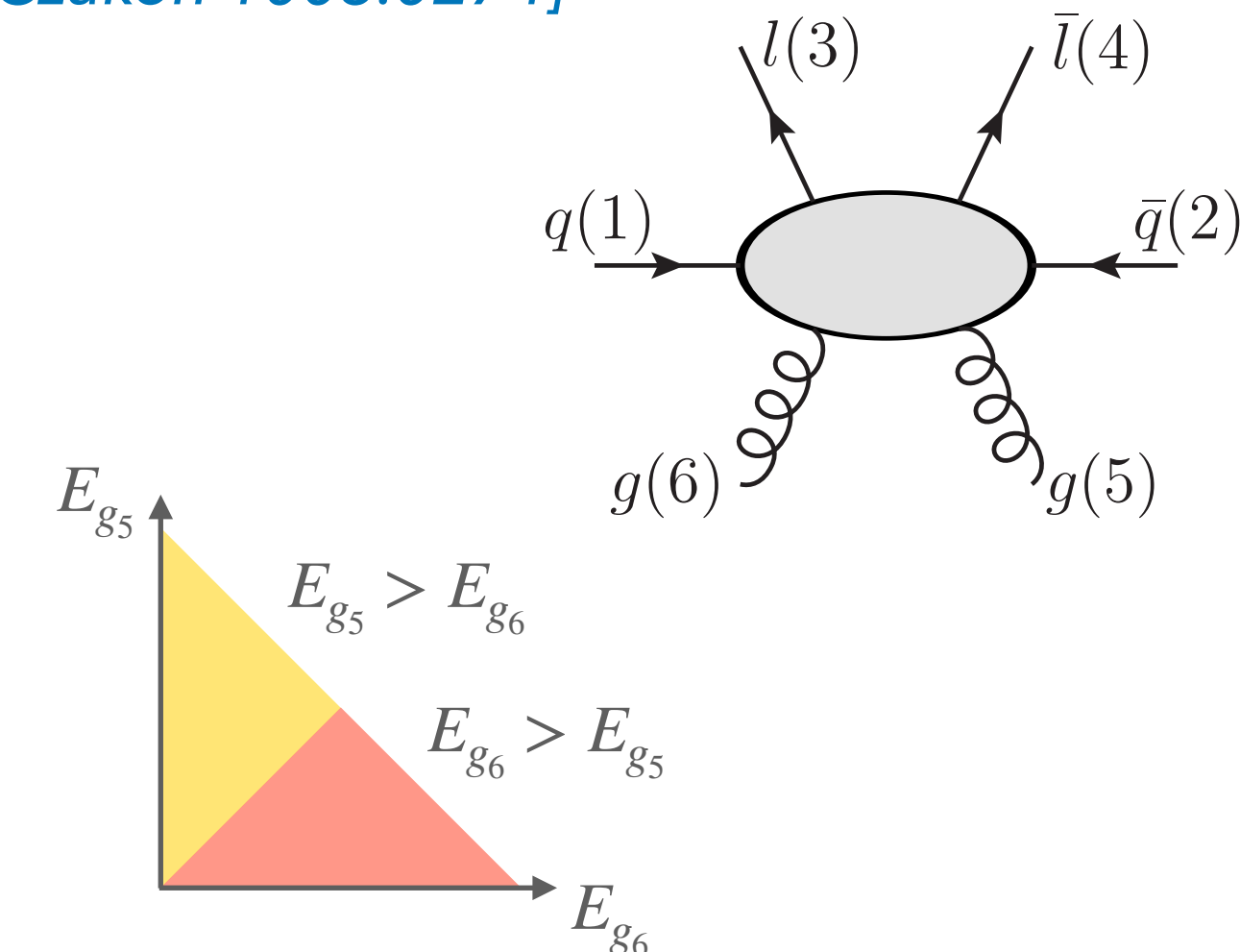
► Exploit colour-coherence: “**nested approach**” → subtract soft limits first, then collinear

► Define subtraction terms in 3 steps:

1. Globally remove soft singularities

► Non-trivial structure of double soft limits

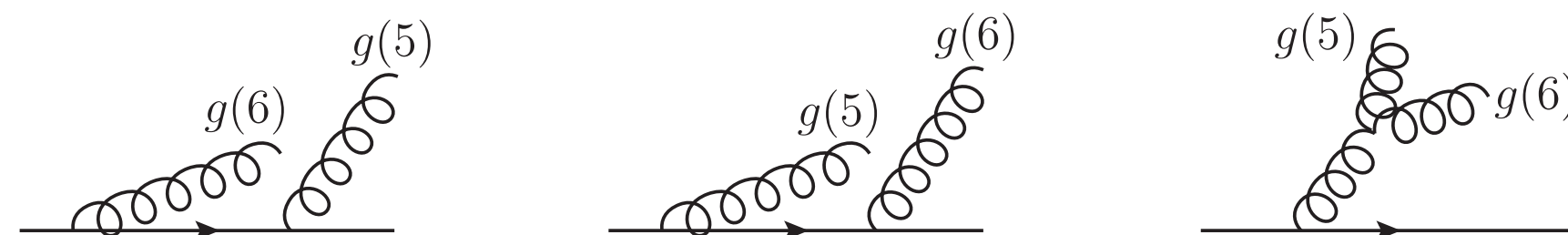
► Strongly-ordered limits to disentangle: energy ordering → $1 = \theta(E_{g_1} - E_{g_2}) + \theta(E_{g_2} - E_{g_1})$



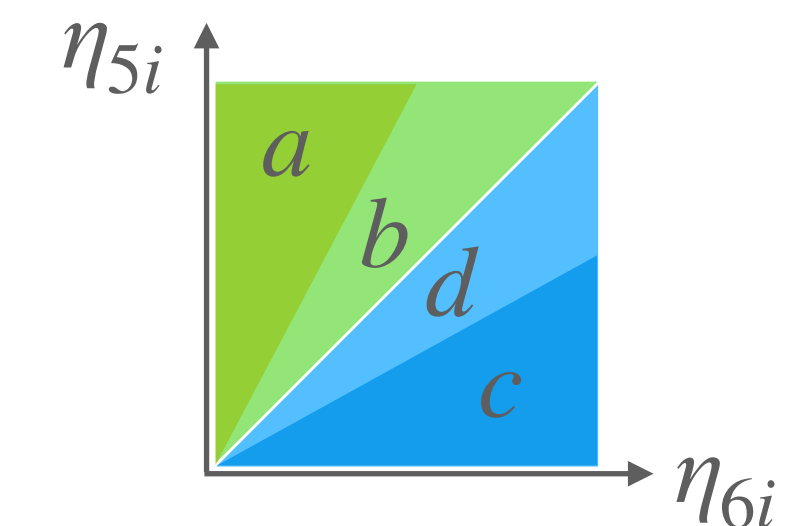
2. FKS partition and sectoring to treat collinear singularities

► Double and triple collinear limits to disentangle → $1 = \sum_{i,j} \omega^{i5,j6}$

► Strongly-ordered limits to disentangle in triple collinear sectors → $\omega^{5i,6i} = \omega^{5i,6i} (\theta_a + \theta_b + \theta_c + \theta_d)$

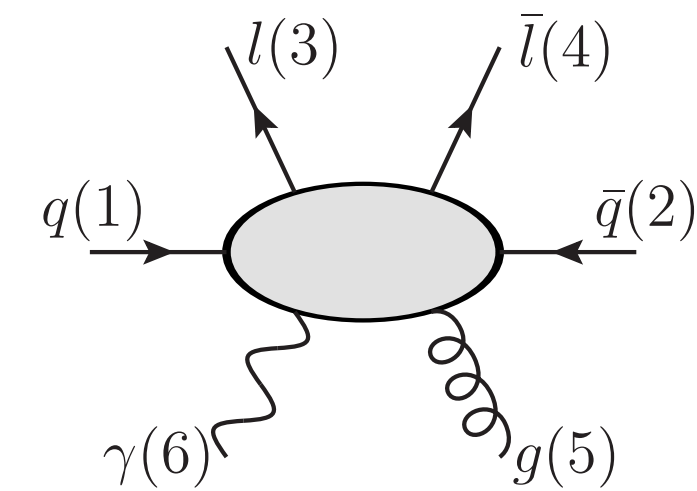
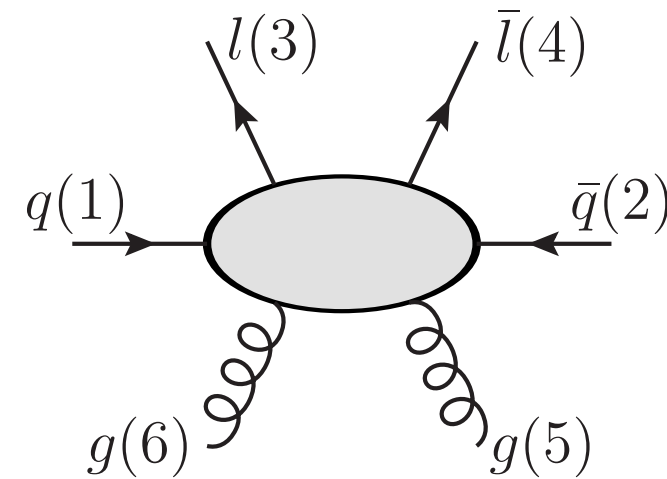


$$\eta_{ab} = \frac{1 - \cos \vartheta_{ab}}{2}$$



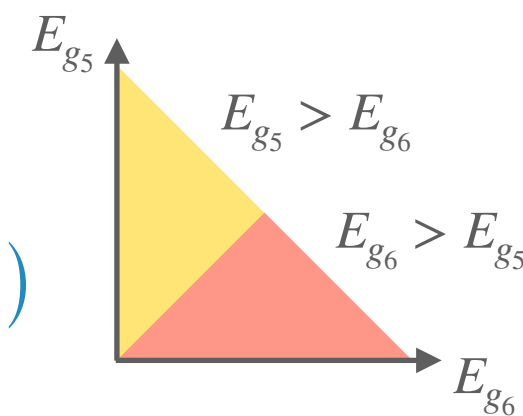
3. Analytic integration of relevant counterterms using reverse unitarity [Anastasiou, Melnikov '02][Caola, Delto, Frellesvig, Melnikov '18, '19]

ADAPTING NSC TO MIXED CORRECTIONS TO DY PROCESS, WHAT CHANGES



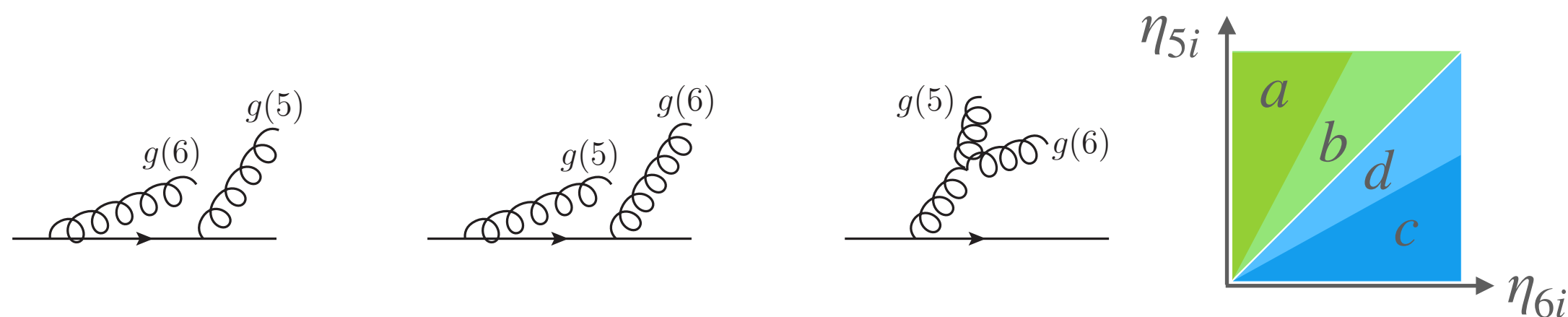
1. Soft singularities

- ▶ Non-trivial structure of double soft limits
- ▶ Energy ordering $\rightarrow 1 = \theta(E_{g_1} - E_{g_2}) + \theta(E_{g_2} - E_{g_1})$



2. Collinear singularities

- ▶ Double and triple collinear partitions: $(4 \text{ TC}) \times 2 + (\text{DC}) \times 2 = 10$
- ▶ Strongly-ordered limits to disentangle in triple collinear sectors $\rightarrow \omega^{5i,6i} = \omega^{5i,6i} (\theta_a + \theta_b + \theta_c + \theta_d)$



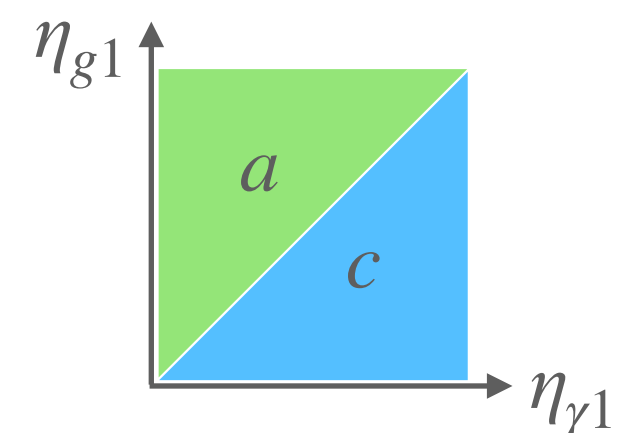
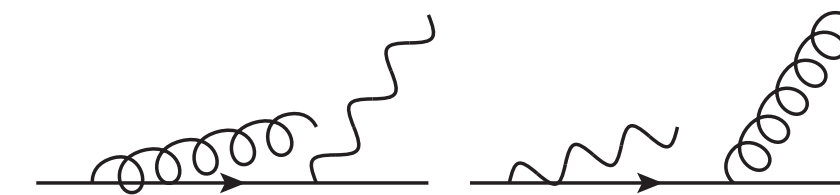
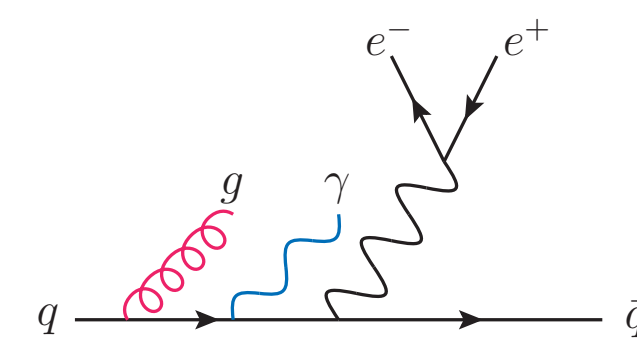
3. Analytic integration of relevant counterterms

1. Soft singularities

- ▶ Double soft limit factorises into **NLO QCD×NLO EW**
 $\rightarrow$ No need for energy ordering

2. Collinear singularities

- ▶ **More "DC" sectors** for FSR: $(2 \text{ TC}) \times 2 + (3 \text{ DC}) \times 2 = 10$
- ▶ No photon-gluon interactions: smaller **partitioning**
 $\rightarrow \omega^{5i,6i} = \omega^{5i,6i} (\theta_a + \theta_c)$



3. Abelianization of QCD results [Delto, Jaquier, Melnikov, Röntsch '19]

NLO EW: WARM UP

► **LO cross section:**

$$2s \, d\hat{\sigma}_{ab}^{\text{LO}} = \mathcal{N} \int \prod_{i=3}^{N_p} [dp_i] F_{\text{LM}}(1_a, 2_b; 3, \dots, N_p; X) = \langle F_{\text{LM}} \rangle$$

$[dp_i] = \frac{d^3 p_i}{(2\pi)^3 2E_i}$

$F_{\text{LM}}(1_a, 2_b; 3, \dots, N_p; X)$

initial-state partons

arbitrary number of FS partons

$\equiv$

$d\text{Lips}_X$

X phase-space measure

$|\mathcal{M}_0|^2$

Tree-level matrix element

$\mathcal{O}$

IR-safe observable

NLO EW: WARM UP

► **LO cross section:**

$$2s \, d\hat{\sigma}_{ab}^{\text{LO}} = \mathcal{N} \int \prod_{i=3}^{N_p} [dp_i] \, F_{\text{LM}}(1_a, 2_b; 3, \dots, N_p; X) = \langle F_{\text{LM}} \rangle$$

$[dp_i] = \frac{d^3 p_i}{(2\pi)^3 2E_i}$

$F_{\text{LM}}(1_a, 2_b; 3, \dots, N_p; X) \equiv \text{dLips}_X \, |\mathcal{M}_0|^2 \, \mathcal{O}$

initial-state partons
arbitrary number of FS partons

Neutral system

X phase-space measure · $\delta(p_a + p_b - \sum_{i=3}^{N_p} p_i - p_X)$

IR-safe observable
Tree-level matrix element

► **NLO EW cross section:**

$$2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = d\hat{\sigma}_{ab}^{\text{V}} + d\hat{\sigma}_{ab}^{\text{R}} + d\hat{\sigma}_{ab}^{\text{pdf}}$$

► Isolation of real-radiation singularities

$$\langle F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle = \langle S_{\mathbf{m}} F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle + \sum_i \langle (1 - S_{\mathbf{m}}) C_{\mathbf{m}i} \omega^{\mathbf{m}i} F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle$$

Soft subtraction term: complete decoupling of the unresolved γ

Hard-collinear subtraction term: decoupling of the unresolved γ

Fully regulated term: numerical integration
 $\mathcal{O}_{\text{NLO}}^{(\mathbf{m})} = \sum (1 - S_{\mathbf{m}}) (1 - C_{\mathbf{m}i}) \omega^{\mathbf{m}i}$

NLO EW: WARM UP

► **LO cross section:** $2s \, d\hat{\sigma}_{ab}^{\text{LO}} = \mathcal{N} \int \prod_{i=3}^{N_p} [dp_i] F_{\text{LM}}(1_a, 2_b; 3, \dots, N_p; X) = \langle F_{\text{LM}} \rangle$

$$[dp_i] = \frac{d^3 p_i}{(2\pi)^3 2E_i}$$

$$F_{\text{LM}}(1_a, 2_b; 3, \dots, N_p; X) \equiv \underbrace{\text{initial-state partons}}_{\text{arbitrary number of FS partons}} \underbrace{\text{Neutral system}} \underbrace{d\text{Lips}_X}_{X \text{ phase-space measure} \cdot \delta(p_a + p_b - \sum_{i=3}^{N_p} p_i - p_X)} \underbrace{|\mathcal{M}_0|^2}_{\text{Tree-level matrix element}} \underbrace{O}_{\text{IR-safe observable}}$$

► **NLO EW cross section:** $2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = d\hat{\sigma}_{ab}^{\text{V}} + d\hat{\sigma}_{ab}^{\text{R}} + d\hat{\sigma}_{ab}^{\text{pdf}}$

► Isolation of real-radiation singularities

$$\langle F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle = \underbrace{\langle S_{\text{m}} F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle}_{\text{Soft subtraction term: complete decoupling of the unresolved } \gamma} + \sum_i \underbrace{\langle (1 - S_{\text{m}}) C_{\text{mi}} \omega^{\text{mi}} F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle}_{\text{Hard-collinear subtraction term: decoupling of the unresolved } \gamma} + \underbrace{\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} F_{\text{LM}}^{ab}(\mathbf{m}_\gamma) \rangle}_{\text{Fully regulated term: numerical integration}}$$

$$\mathcal{O}_{\text{NLO}}^{(\text{m})} = \sum (1 - S_{\text{m}}) (1 - C_{\text{mi}}) \omega^{\text{mi}}$$

► Final result in the partonic c.o.m frame (real correction + virtual corrections in Catani's scheme + PDFs renormalisation)

$$2s_{q\bar{q}'} \, d\hat{\sigma}_{q\bar{q}'}^{\text{NLO, EW}} = \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} F_{\text{LM}}^{q\bar{q}'}(\mathbf{m}_\gamma) \rangle + \langle F_{\text{LV, fin}}^{q\bar{q}', \text{EW}} \rangle + \frac{\alpha_{\text{ew}}}{2\pi} \left\{ \langle Q_q^2 \tilde{P}_{qq}^{\text{NLO}} \otimes F_{\text{LM}}^{q\bar{q}'} + Q_{q'}^2 F_{\text{LM}}^{q\bar{q}'} \otimes \tilde{P}_{qq}^{\text{NLO}} + \mathcal{G}_{\text{EW}}^W F_{\text{LM}}^{q\bar{q}'} \rangle \right\}$$

- ✓ Pole cancellation proven analytically
- ✓ Simple structures arise, compact expressions
- ✓ Fully differential

MAIN MESSAGES

- ▶ Mixed QCD×EW corrections to the Drell-Yan process are important to increase our control on SM parameters and search for new physics
- ▶ Theoretical calculations are reaching the required level of sophistication to provide prediction at sub-percent accuracy. Crucial role played by developments in QCD beyond NLO → importance of cross-checks among different methods
- ▶ Flexibility of existing analytical tools fundamental in avoiding a 'from-scratch' approach → previous studies of NCDY provides a robust framework and most of the required ingredients for subtraction

FINITE REMAINDER FOR CCDY

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} =$$

$\vdots$
 $\bullet$
double-virtual

 $\vdots$
 $\bullet$
real-virtual

 $\vdots$
 $\bullet$
double-real

 $\vdots$
 $\bullet$
coll. renor.

- ✓ **Pole cancellation** proven analytically
- ✓ **Fully differential**
- ✓ **Fully local** → leads to a very stable numerical evaluation
- ✓ **Final result fully analytic**, but cumbersome in generic frame
- ✓ In the **com** reference **frame**, $E_1 = E_2 = E_c$, **the result simplifies** remarkably
- ✓ Simple structures arise, **compact expressions**
- ✓ **Few main kinematic building blocks** can be easily identified

FINITE REMAINDER FOR CCDY

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = d\hat{\sigma}_{ab}^{\text{SU}} + d\hat{\sigma}_{ab}^{\text{DU}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

double-virtual real-virtual double-real coll. renor.

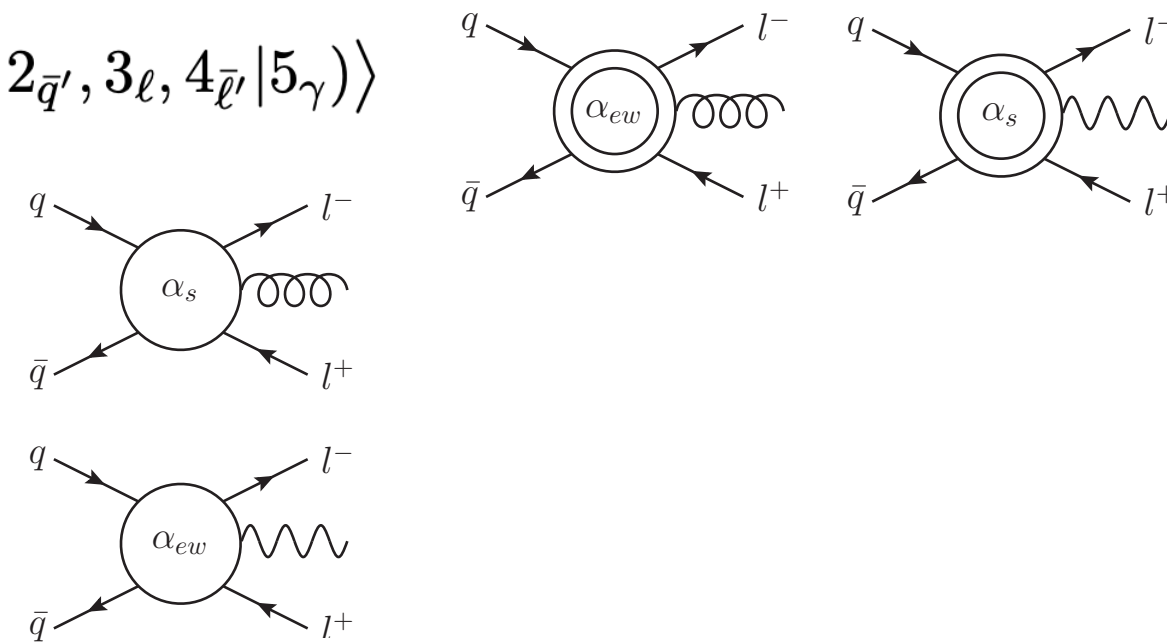
single-unr. double-unr. fully-resolved

fully-resolved = 4-body kinematics
 single-unresolved = 3-body kinematics
 double-unresolved = 2-body kinematics

$$2s \cdot d\hat{\sigma}_{q\bar{q}'}^{\text{SU, el}} = \langle \mathcal{O}_{\text{nlo}}^g F_{\text{RV, fin}}^{(\text{EW})}(1_q, 2_{\bar{q}'}, 3_\ell, 4_{\bar{\ell}'} | 5_g) \rangle + \langle \mathcal{O}_{\text{nlo}}^\gamma F_{\text{RV, fin}}^{(\text{QCD})}(1_q, 2_{\bar{q}'}, 3_\ell, 4_{\bar{\ell}'} | 5_\gamma) \rangle$$

$$+ [\alpha_{\text{ew}}] \langle \mathcal{O}_{\text{nlo}}^g [\mathcal{G}_{\text{nlo}}^{(\text{EW})} F_{\text{LM}}(1_q, 2_{\bar{q}'}, 3_\ell, 4_{\bar{\ell}'} | 5_g)] \rangle$$

$$+ [\alpha_s] \langle \mathcal{O}_{\text{nlo}}^\gamma [C_F \mathcal{G}_{\text{nlo}}^{(\text{QCD})} F_{\text{LM}}(1_q, 2_{\bar{q}'}, 3_\ell, 4_{\bar{\ell}'} | 5_\gamma)] \rangle$$



$$\mathcal{G}_{\text{nlo}}^{(\text{QCD})} = \frac{2\pi^2}{3} + 6L_c + 3 \log \eta_{12} + 2\text{Li}_2(1 - \eta_{12})$$

$$\mathcal{G}_{\text{nlo}}^{(\text{EW})} = Q_\ell^2 A_\ell + Q_\ell Q_q A_{\ell q}(\eta_{13}) + Q_\ell Q_{\bar{q}'} A_{\ell q}(\eta_{23}) + Q_q Q_{\bar{q}'} A_{qq}(\eta_{12})$$

$$A_\ell = \frac{13}{2} - \frac{2\pi^2}{3} + 2L_3^2 - 2L_c^2 - 3L_3$$

$$A_{\ell q}(\eta) = \frac{\pi^2}{3} + L_3^2 - 3L_c^2 + L_3(-3 + 2L_c + 2\log \eta) - L_c(3 + 2\log \eta) - 3\log \eta - 2\text{Li}_2(1 - \eta)$$

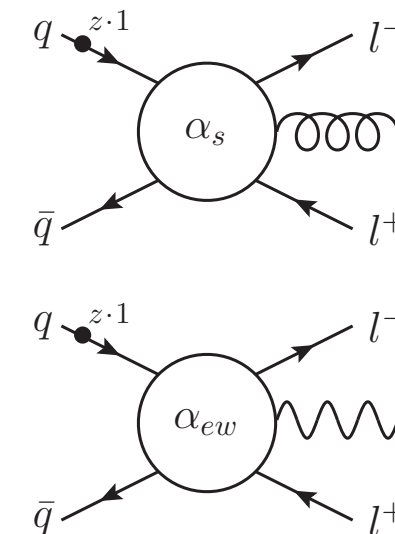
$$A_{qq}(\eta) = -\frac{2\pi^2}{3} - 6L_c - 3\log \eta - 2\text{Li}_2(1 - \eta)$$

$$2s \cdot d\hat{\sigma}_{q\bar{q}'}^{\text{SU, sb}} = [\alpha_{\text{ew}}] \langle Q_q^2 \mathcal{O}_{\text{nlo}}^g [\tilde{P}_{qq}^{\text{NLO}}(z, E_c) + \tilde{\omega}_{\gamma||1}^{\gamma 1, g1} \log \eta_{15} \bar{P}_{qq, R}^{\text{AP}, 0}(z)] \otimes F_{\text{LM}}(1_q, 2_{\bar{q}'}, 3_\ell, 4_{\bar{\ell}'} | 5_g) \rangle + (1 \leftrightarrow 2)$$

$$+ [\alpha_s] \langle C_F \mathcal{O}_{\text{nlo}}^\gamma [\tilde{P}_{qq}^{\text{NLO}}(z, E_c) + \tilde{\omega}_{g||1}^{\gamma 1, g1} \log \eta_{15} \bar{P}_{qq, R}^{\text{AP}, 0}(z)] \otimes F_{\text{LM}}(1_q, 2_{\bar{q}'}, 3_\ell, 4_{\bar{\ell}'} | 5_\gamma) \rangle + (1 \leftrightarrow 2)$$

$$\tilde{P}_{qq}^{\text{NLO}}(z, E) = 4\mathcal{D}_1(z) - 2(1+z)\log(1-z) + (1-z) + \bar{P}_{qq, R}^{\text{AP}, 0}(z) \log\left(\frac{4E^2}{\mu^2}\right)$$

$$\bar{P}_{qq, R}^{\text{AP}, 0}(z) = 2\mathcal{D}_0(z) - (1+z),$$



$$2s \cdot d\hat{\sigma}_{q\bar{q}'}^{\text{FS}} = \langle (I - S_g)(I - S_\gamma) \Omega_1^{q\bar{q}'} F_{\text{LM}}(1_q, 2_{\bar{q}'}, 3_\ell, 4_{\bar{\ell}'} | 5_g, 6_\gamma) \rangle$$

$$\Omega_1^{q\bar{q}'} = (1 - C_{g\gamma, 1})(1 - C_{g1}) \omega^{\gamma 1, g1} \theta_a^{(1)} + (1 - C_{g\gamma, 1})(1 - C_{\gamma 1}) \omega^{\gamma 1, g1} \theta_b^{(1)}$$

$$+ (1 - C_{g\gamma, 2})(1 - C_{g2}) \omega^{\gamma 2, g2} \theta_a^{(2)} + (1 - C_{g\gamma, 2})(1 - C_{\gamma 2}) \omega^{\gamma 2, g2} \theta_b^{(2)}$$

$$+ (1 - C_{g2})(1 - C_{\gamma 1}) \omega^{\gamma 1, g2} + (1 - C_{g1})(1 - C_{\gamma 2}) \omega^{\gamma 2, g1}$$

$$+ (1 - C_{g2})(1 - C_{\gamma 3}) \omega^{\gamma 3, g2} + (1 - C_{g2})(1 - C_{\gamma 4}) \omega^{\gamma 4, g2}$$

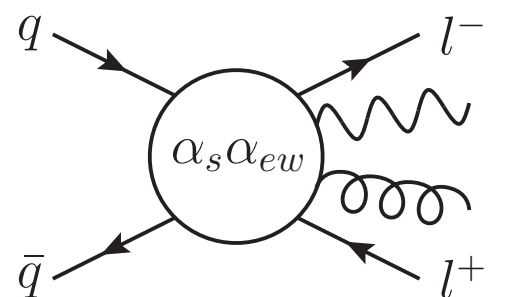
$$+ (1 - C_{g1})(1 - C_{\gamma 3}) \omega^{\gamma 3, g1} + (1 - C_{g1})(1 - C_{\gamma 4}) \omega^{\gamma 4, g1},$$

► Fully resolved: **computed numerically**

► Sector functions $\theta_{(i=a,b)}$ absorbed in the parametrisation [Czakon '10+'11]

► Each sector is treated separately and parallelised

► In each sector and partition, at most **7 different kinematic configurations**

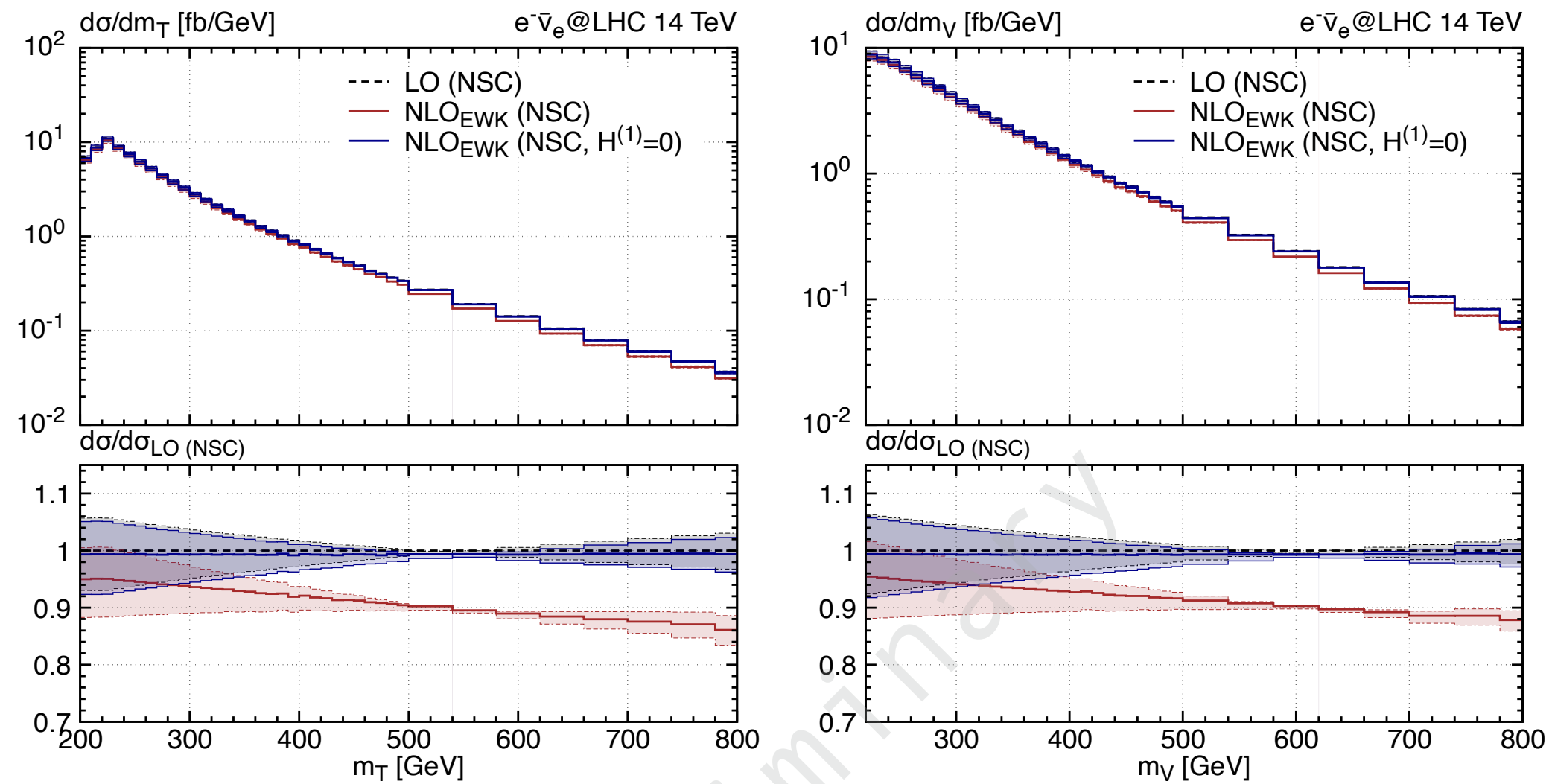


MAIN MESSAGES

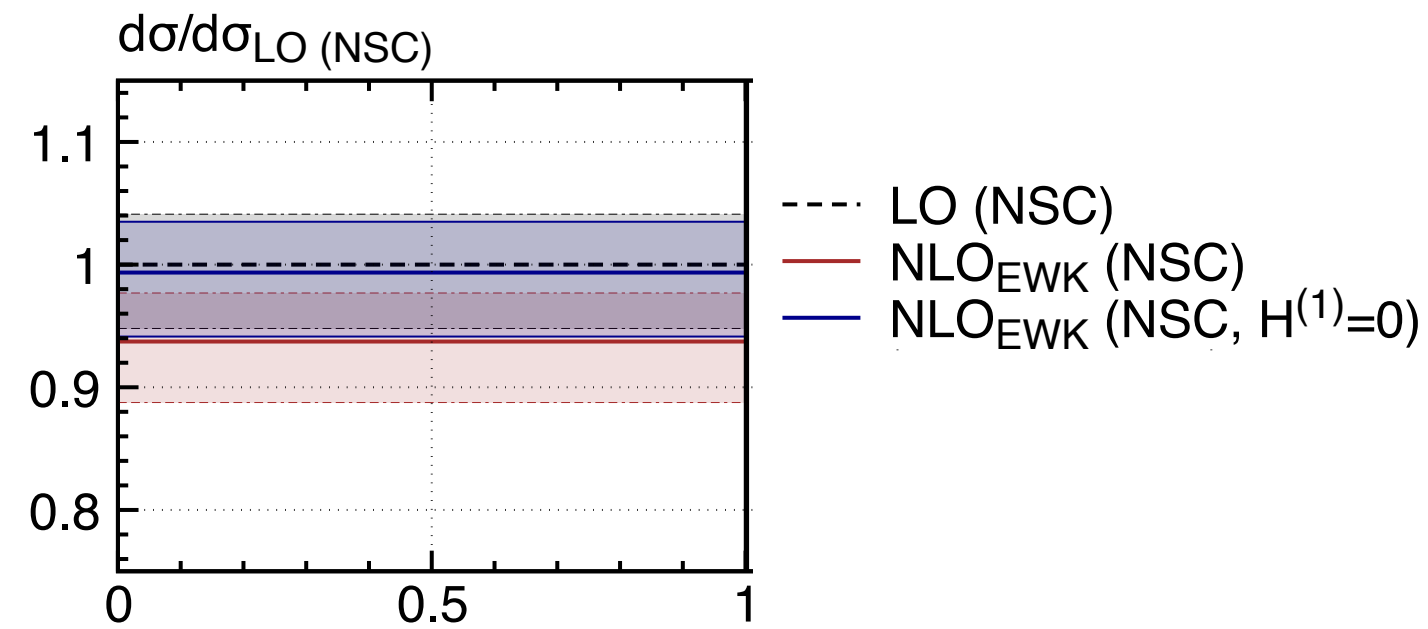
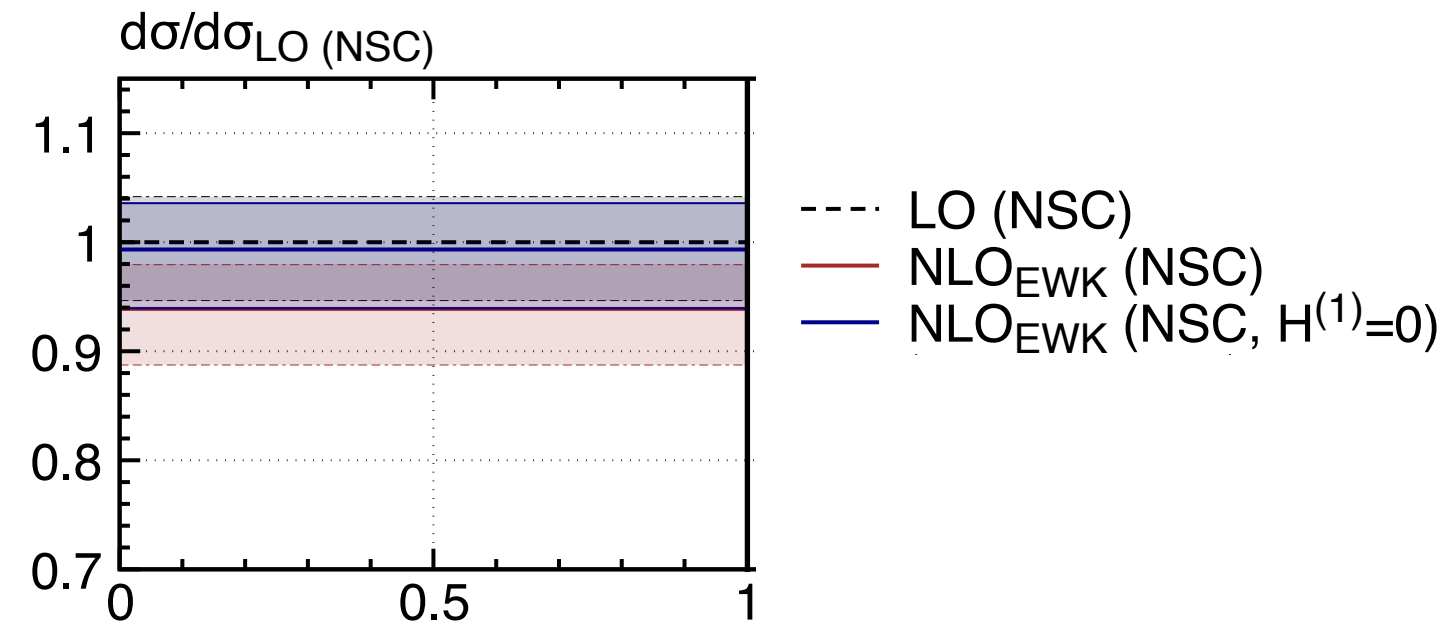
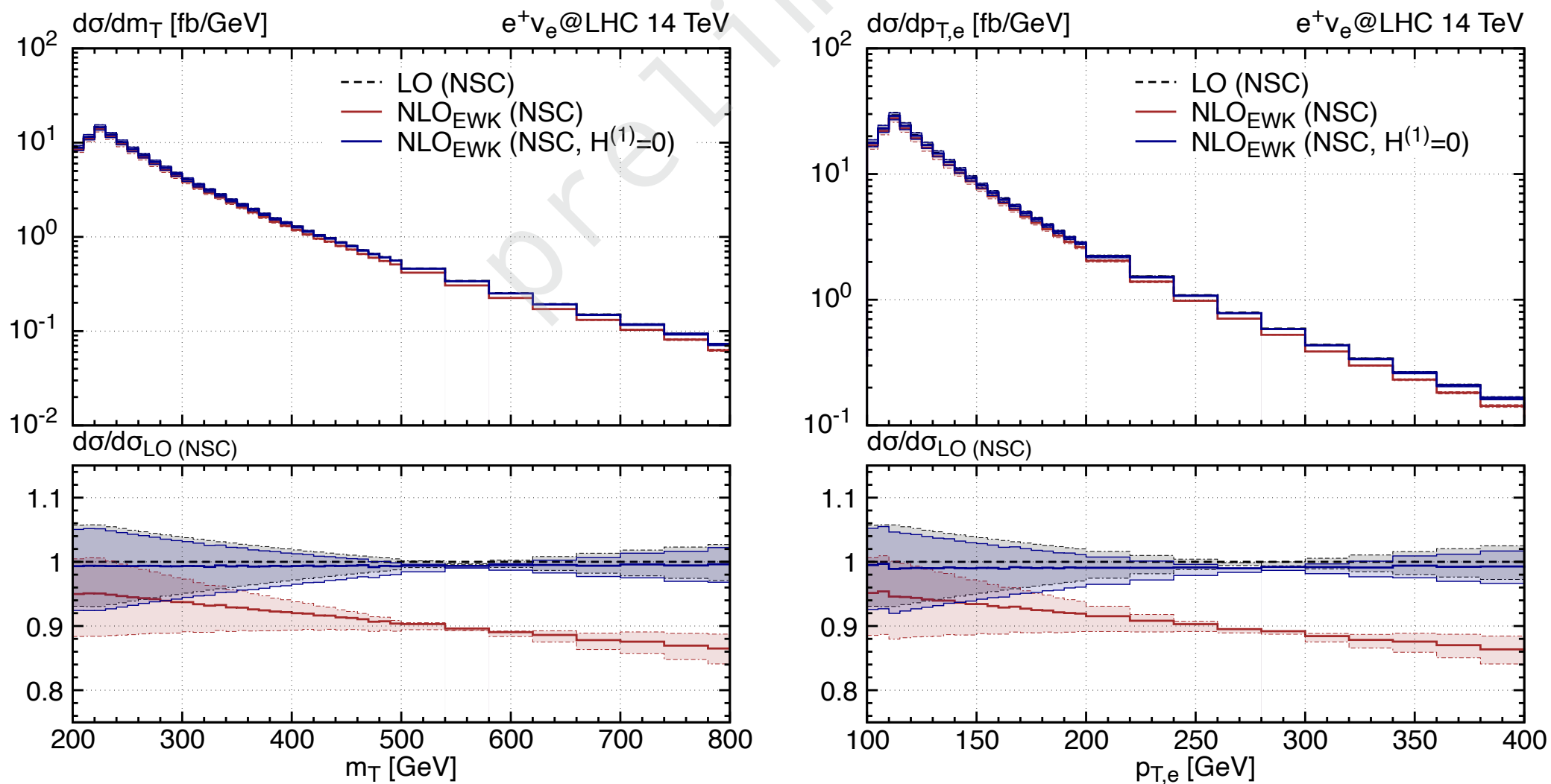
- ▶ Mixed QCD×EW corrections to the Drell-Yan process are important to increase our control on SM parameters and search for new physics
- ▶ Theoretical calculations are reaching the required level of sophistication to provide prediction at sub-percent accuracy. Crucial role played by developments in QCD beyond NLO → importance of cross-checks among different methods
- ▶ Flexibility of existing analytical tools fundamental in avoiding a 'from-scratch' approach → previous studies of NCDY provides a robust framework and most of the required ingredients for subtraction
- ▶ Flexibility of existing numerical tools (and foreseen generalisations) fundamental to obtain reliable results in limited amount of time

NLO EW

CCDY-



CCDY+

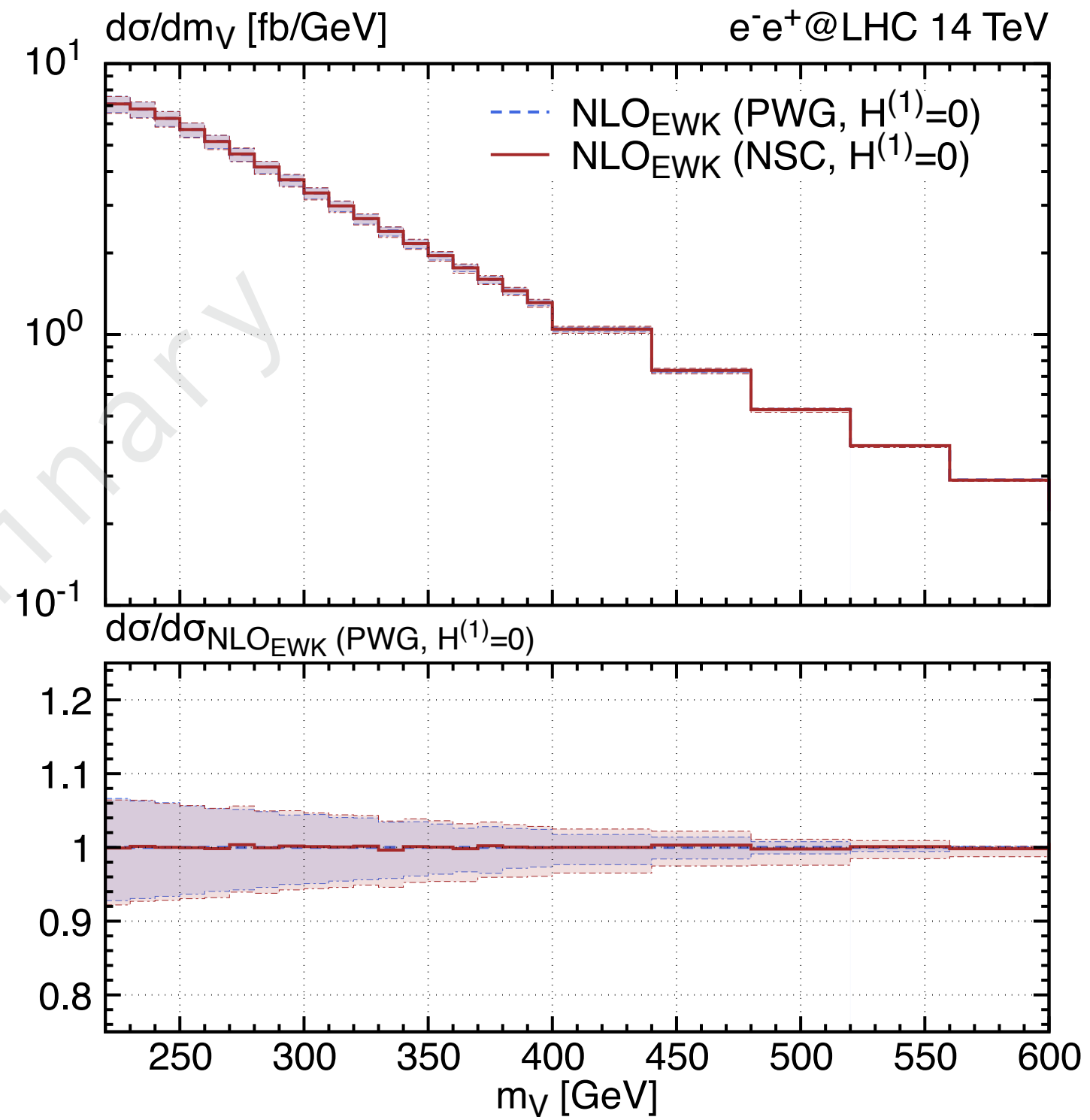
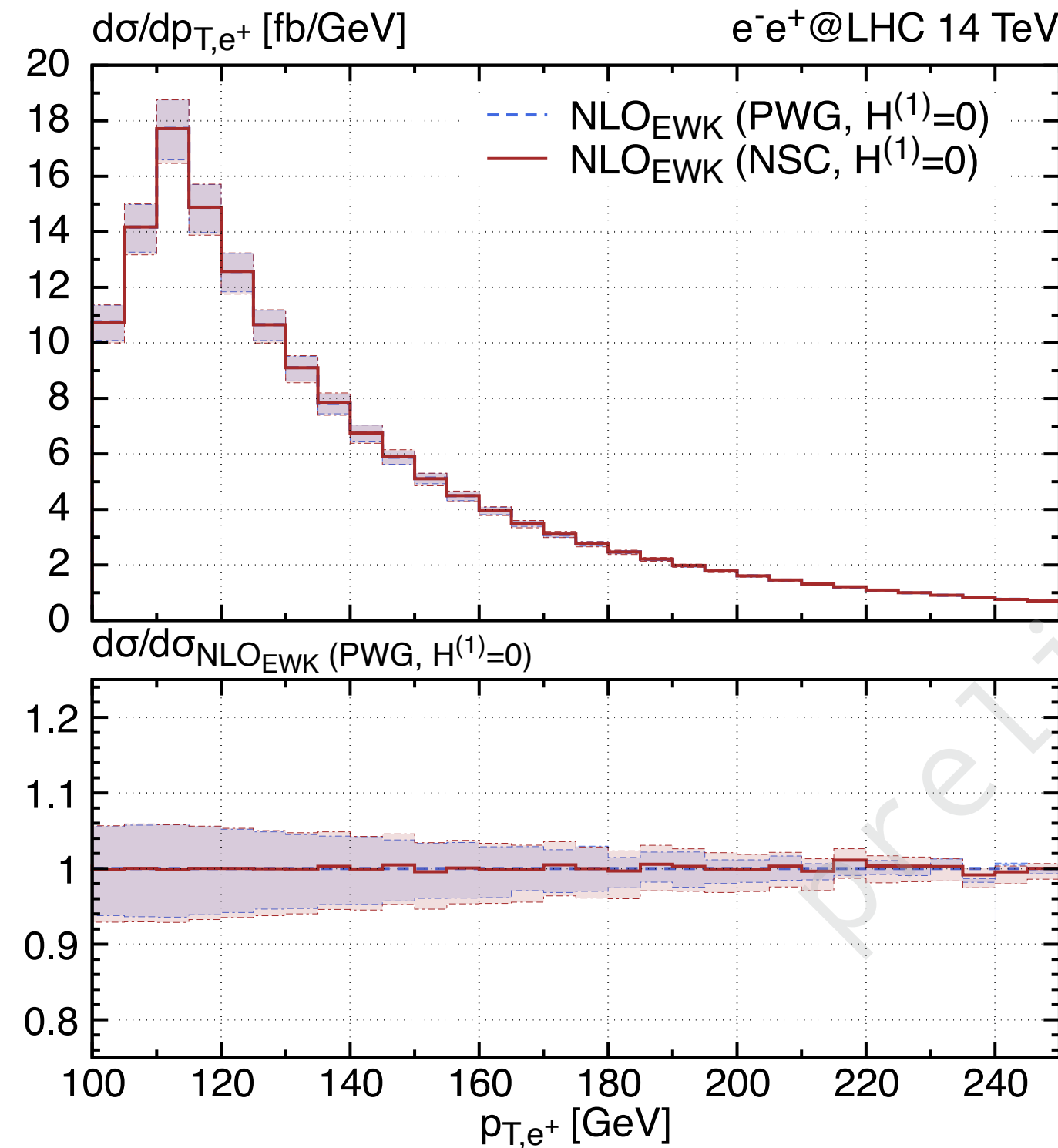


- ▶ NLO EW corrections show the expected behaviour
- ▶ The impact of the virtual component is about 3% at integrated level
- ▶ At differential level, virtual corrections grow by about 10% in the transverse mass window [200-800]GeV
- ▶ Compatible with the behaviour of Sudakov logarithms

$H(1)$ = finite remainder of 1-loop squared amplitudes

NLO EW

NCDY

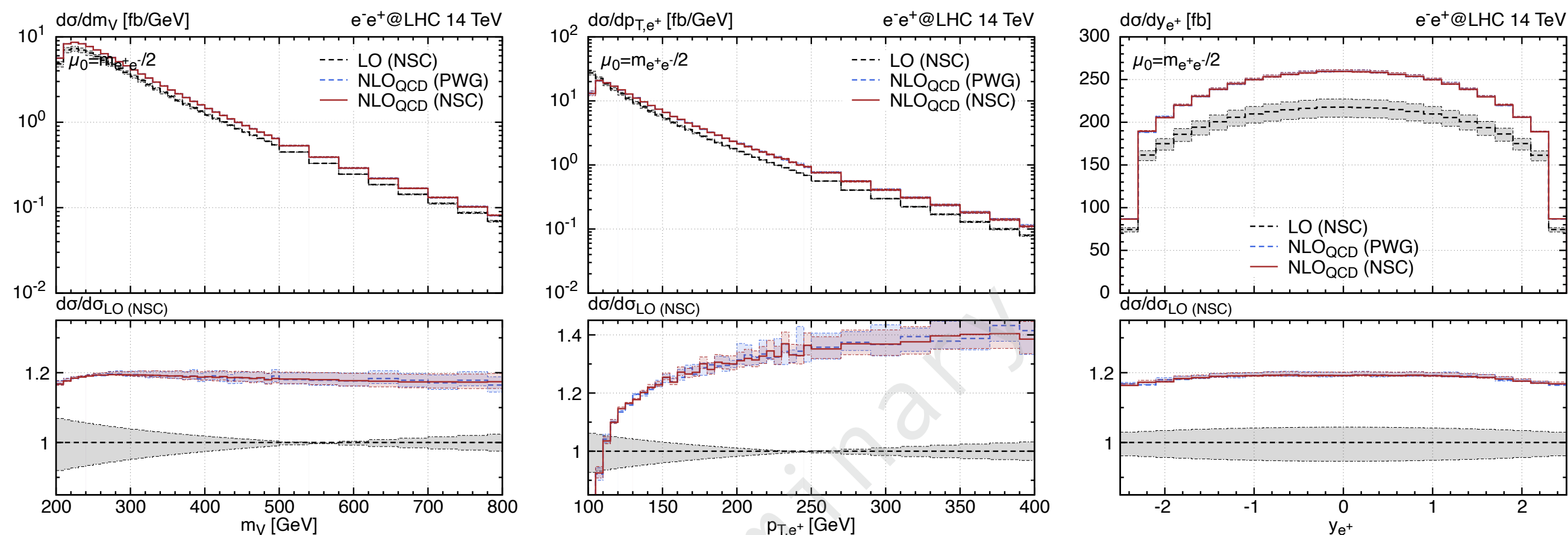


- ▶ NLO EW corrections show the expected behaviour
- ▶ The impact of the virtual component is about 3% at integrated level
- ▶ At differential level, virtual corrections grow by about 10% in the transverse mass window [200-800]GeV
- ▶ Compatible with the behaviour of Sudakov logarithms
- ▶ Independent validation against POWHEG ongoing

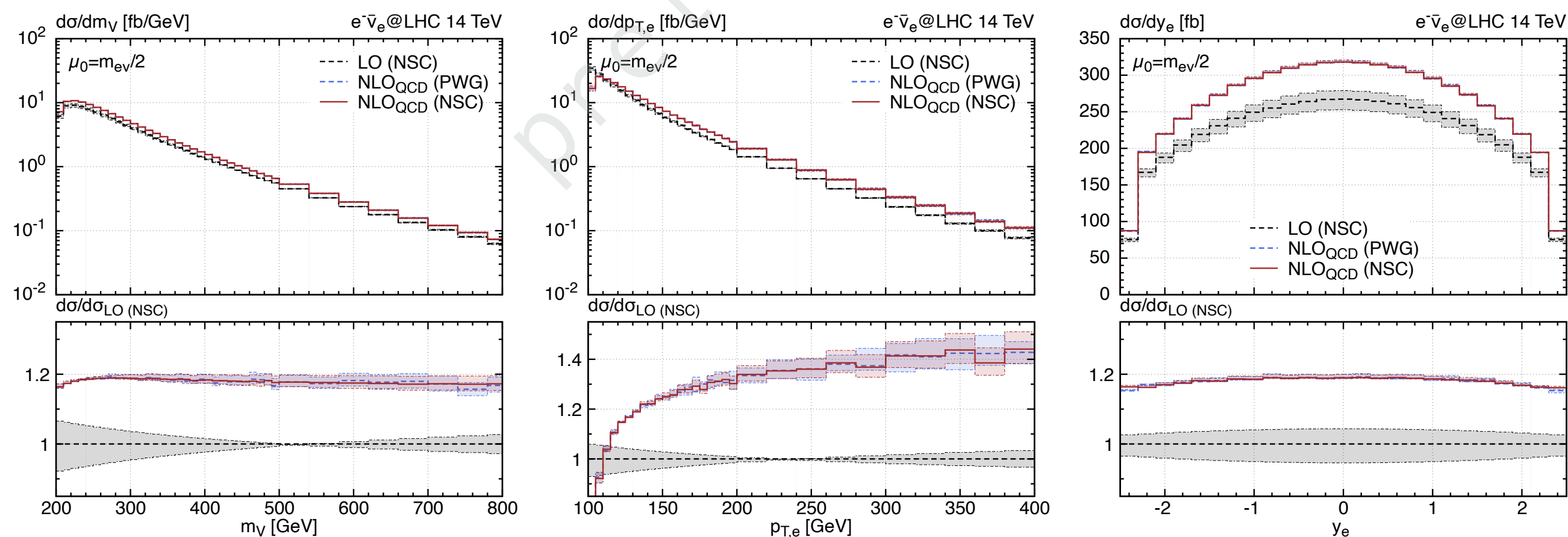
$H(1)$ = finite remainder of 1-loop squared amplitudes

VALIDATION @NLO QCD: BENCHMARKS FROM POWHEG

NCDY



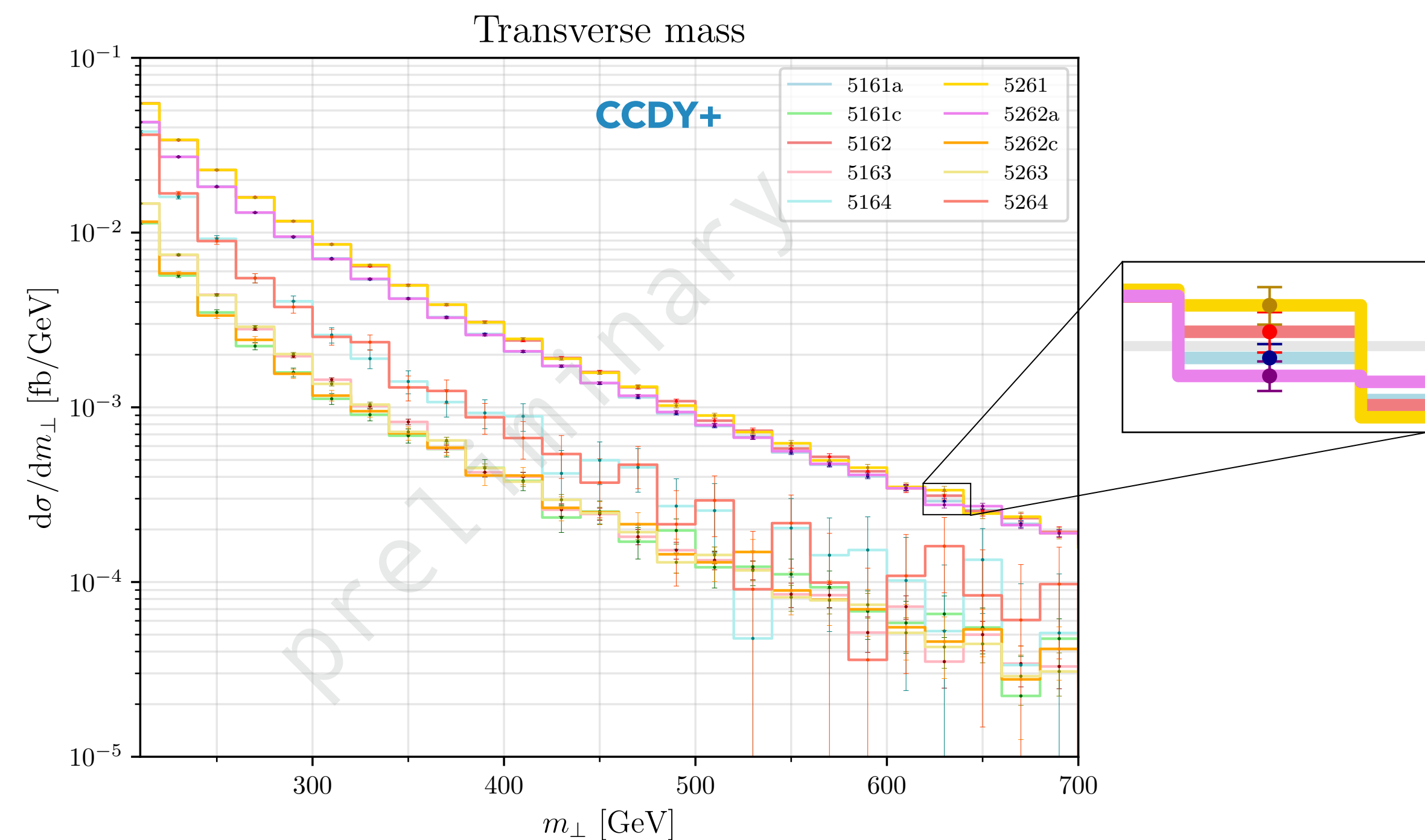
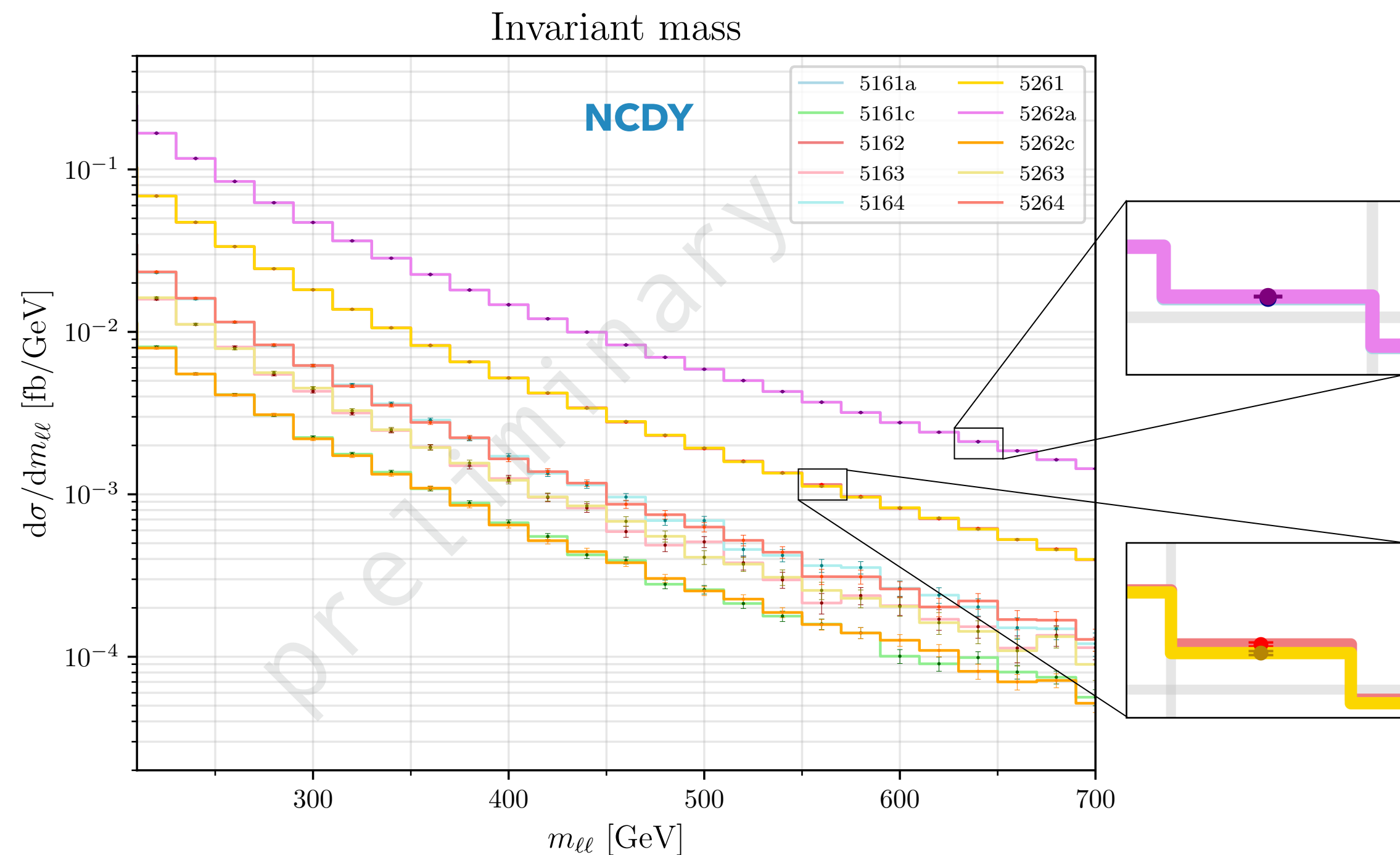
CCDY-



- ▶ Perfect agreement between a completely independent implementation in POWHEG and NSC (single code for CCDY and NCDY).
[process set by pre-process flag: minimising routines duplication and reducing room for bugs]
- ▶ NLO QCD corrections large and far from being captured by the LO scale variation.
- ▶ Importance of a proper scale choice to better estimate the missing higher order terms. Different options are under inspection
- ▶ NSC match the same level of accuracy of POWHEG with $\sim 10^{-3}$ run time*
[not really a fair comparison as PWG is not designed for FO]

*NSC on-the-fly run on a single core, single seed, for 13min.
PWG 64-core 64-parallel jobs with an average of 1h per core.

MIXED QCD×EW CORRECTIONS: NCDY VS CCDY (DOUBLE-REAL SUBTRACTION)



- ▶ Z: sectors “come in pairs”; the symmetry of the process enforces similarities between sectors that have analogous singular structure*

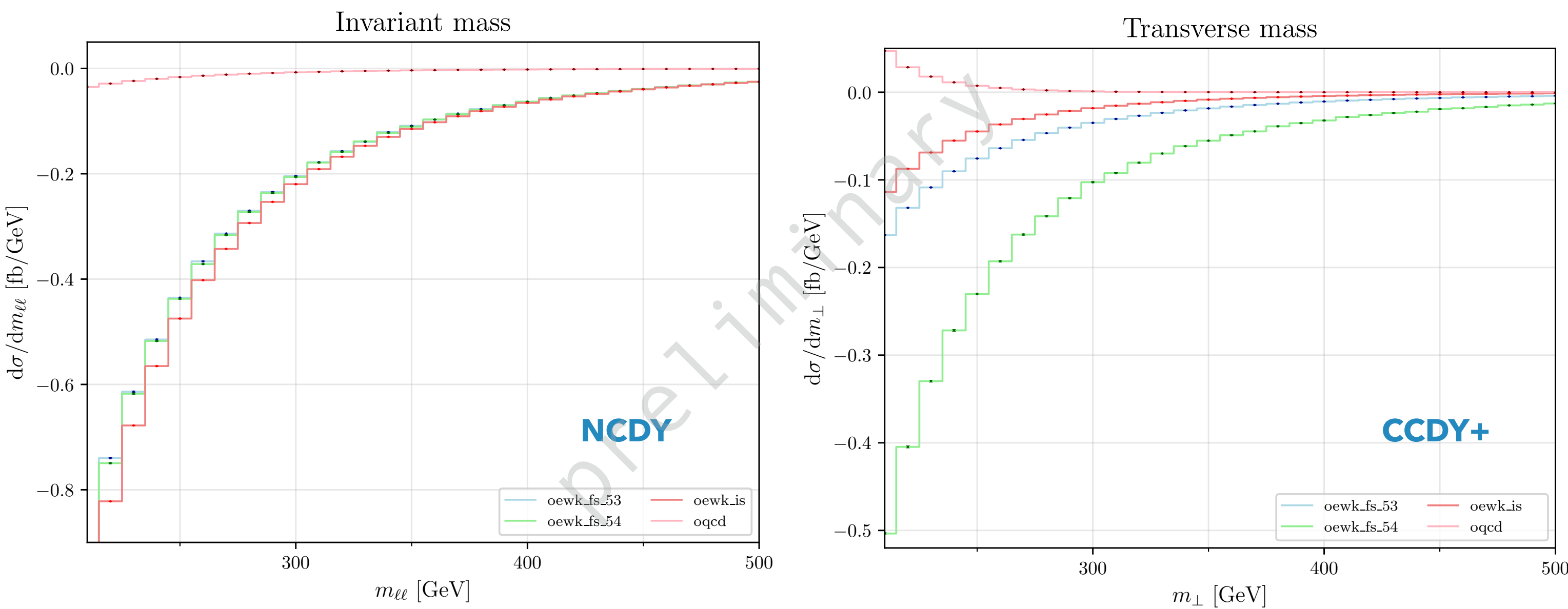
- ▶ Sectors that are more unstable also do count less

- ▶ W: the asymmetry within initial states and final state and their interplay is reflected in a less pronounced difference between sectors*

- ▶ Hierarchy between triple collinear sector and double collinear sectors different in the two cases.

*details in the backup (computing cost per each plot ~ 70000 CPUh)

MIXED QCD×EW CORRECTIONS: NCDY VS CCDY (REPRESENTATIVE SINGLE UNRESOLVED CONTRIBUTIONS)



- CCDY:
- ▶ reduced EW magnitude
 - ▶ better precision in key sectors
 - ▶ strong asymmetries not present in NCDY
- NCDY:
- ▶ more symmetric
 - ▶ but dominated by larger cancellations

$\mathcal{O}_{\text{nlo}}$ contributions: NCDY vs CCDY (W^+)					
(100 seeds, $5.0 \times 10^5 \times 15 + 5.0 \times 10^6 \times 10$)					
NCDY			CCDY (W^+)		
Sector	Runtime	Cross-section	Sector	Runtime	Cross-section
oqcd	0:43:29	-0.27737 ± 0.00029	oqcd	5:31:26	0.20859 ± 0.00008
oewk_fs_53	0:33:16	-7.56029 ± 0.00214	oewk_fs_53	6:03:29	-1.32523 ± 0.00015
oewk_fs_54	0:32:28	-7.61950 ± 0.00252	oewk_fs_54	4:48:14	-4.04229 ± 0.00139
oewk_is	0:52:54	-8.21967 ± 0.00069	oewk_is	8:36:13	-0.77722 ± 0.00019

- ▶ CCDY: EW contributions significantly smaller in absolute value than NCDY (fs_53 reduced by $\sim 1/6$, fs_54 reduced by $\sim$ factor 1/2)
- ▶ QCD soft term flips sign (suggests different interplay of subtraction vs Born structure)
- ▶ CCDY runs are $\sim 5\text{--}6\text{h}$, vs $\sim 30\text{--}50'$ for NCDY (integrand is more complex or less efficient)
- ▶ Different symmetry between sectors (as expected by flavour structure)
 - ▶ NCDY: fs_53 $\approx$ fs_54 (very symmetric)
 - ▶ CCDY: fs_53 ≈ -1.33 , fs_54 ≈ -4.04 (factor ~ 3 difference)

MAIN MESSAGES

- ▶ Mixed QCD×EW corrections to the Drell-Yan process are important to increase our control on SM parameters and search for new physics
- ▶ Theoretical calculations are reaching the required level of sophistication to prediction at sub-percent accuracy. Crucial role played by developments in QCD beyond NLO → importance of cross-checks among different methods
- ▶ Flexibility of existing analytical tools fundamental in avoiding a 'from-scratch' approach → previous studies of NCDY provides a robust framework and most of the required ingredients for subtraction
- ▶ Flexibility of existing numerical tools (and foreseen generalisations) fundamental to obtain reliable results in limited amount of time
- ▶ Generalising QCD×EW finite remainder [from subtraction procedure] to arbitrary processes should be possible by abelianising fully general QCD results

ABELIANISING THE GENERAL QCD RESULT

1. Consider directly the **final expression of NLO QCD correction** [Devoto et al '24, '25]

$$2s_{ab} d\hat{\sigma}_{ab}^{\text{NLO}} = \sum_n \left\langle \mathcal{O}_{\text{NLO}}^{(\text{m})} \Delta^{(\text{m})} \left[F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m}_g)] + \sum_{\rho=1}^{n_f} \left[F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m}_{q_\rho})] + (q_\rho \mapsto \bar{q}_n) \right] \right] \right\rangle$$

$$+ \sum_n \left[[\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \rangle + \langle F_{\text{LV,fin}}^{ab}[\mathcal{B}_{N,n}] \rangle \right] + \sum_x \sum_n [\alpha_s] \left\langle \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N,n}] + F_{\text{LM}}^{ax}[\mathcal{B}_{N,n}] \otimes \mathcal{P}_{xb}^{\text{NLO}} \right\rangle$$

See D. M. Tagliabue's talk

2. **Abelianise** the result by applying the replacements [for $q_1 + q_2' \rightarrow \ell_3 + \ell_4' : N = 4, n = 1, n_f = 0$]

- $\mathbf{m}_g \mapsto \mathbf{m}_\gamma, [\alpha_s] \mapsto [\alpha_{\text{ew}}]$: being $[\alpha_{\text{ew}}]$ defined analogously of its strong counterpart.
- $\mathbf{T}_i \cdot \mathbf{T}_j \mapsto Q_i Q_j$: being scalars, the electric charges $Q_{i(j)}$ always commute with each other.
- $C_A \mapsto 0$: we retain only the quark-gluon interactions.
- $\mathbf{T}_i^2 = C_{f_i} \mapsto Q_i^2$: having removed all the C_A factors in the previous steps, here we only have to consider C_{F}^2 . Such replacement has to be performed also in the collinear anomalous dimensions.
- $\mathbf{T}_1 \mapsto -Q_1, \mathbf{T}_1 \mapsto -Q_2, \mathbf{T}_3 \mapsto Q_3, \mathbf{T}_4 \mapsto Q_4$: the signs are a remnants of Catani's convention (all partons are considered to be outgoing).

$\Rightarrow$

$$2s_{q_1 q_2} d\hat{\sigma}_{q_1 q_2}^{\text{NLO}} = \langle \mathcal{O}_{\text{NLO}}^{(\text{m})} F_{\text{LM}}^{q_1 q_2}[\mathcal{B}_4(\mathbf{m}_\gamma)] \rangle + \langle F_{\text{LV,fin}}^{q_1 q_2}[\mathcal{B}_4] \rangle$$

$$+ \sum_{i,j>i=1}^4 [\alpha_{\text{ew}}] \langle \mathcal{G}_{ij}^{\text{ew}} F_{\text{LM}}^{q_1 q_2}[\mathcal{B}_4] \rangle + \sum_{i=1}^4 [\alpha_{\text{ew}}] \langle \mathcal{G}_i^{\text{ew}} F_{\text{LM}}^{q_1 q_2}[\mathcal{B}_4] \rangle$$

$$+ [\alpha_{\text{ew}}] \left\langle Q_1^2 \mathcal{P}_{q_1 q_1}^{\text{NLO}} \otimes F_{\text{LM}}^{q_1 q_2}[\mathcal{B}_4] + Q_2^2 F_{\text{LM}}^{q_1 q_2}[\mathcal{B}_4] \otimes \mathcal{P}_{q_2 q_2}^{\text{NLO}} \right\rangle,$$

NCDY: $Q_1 = -Q_2, Q_3 = -Q_4$
CCDY: $Q_1 + Q_2 = Q_W = Q_{3(4)}$

3. The results can be obtained without having to recompute anything

$$\mathcal{G}_{ij}^{\text{ew}} = Q_i Q_j \left\{ L_{ij}^2 - 2 \left(2L_{\text{max}} + \frac{1}{2} \log \eta_{ij} \right) \log \eta_{ij} - 2 \text{Li}_2(1 - \eta_{ij}) + L_{ij} \left(\frac{\gamma_i}{T_{2,i}} + \frac{\gamma_j}{T_{2,j}} \right) - \pi^2 \lambda_{i,j} \right\}$$

$$\mathcal{G}_i^{\text{ew}} = Q_i^2 \left[2L_{\text{max}}^2 - \frac{\pi^2}{6} - \left(2\tilde{L}_i \gamma_i^{22,(0)} - \gamma_i^{22,(1)} \right) \frac{\theta_{\mathcal{H}_f}}{Q_i^2} - 2 \left(L_i^2 + 2L_i \tilde{L}_i + \tilde{L}_i \frac{\gamma_i}{Q_i^2} \right) \bar{\theta}_{\mathcal{H}_f} \right]$$

ABELIANISING THE GENERAL QCD RESULT

At **NNLO** things get more complicated:

1. The general results for NNLO QCD is based on energy ordering and sum over channels
2. In some cases the ordering is easy to reverse, thanks to the dependence over $E_{\max}$, in some cases it requires more care

Example:

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{SU,el}} = & \sum_{\mathbf{m}} \sum_n \left\{ [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} I_T^{(0)}(E_{\mathbf{m}}) + \bar{\delta}_{\mathbf{m}g} I_T^{(0)}(E_{\max})] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \right. \\
 & - \sum_{i \in \mathcal{H}} [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{i||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \log\left(\frac{\eta_{i\mathbf{m}}}{2}\right) \Delta^{(\mathbf{m})} \left[\delta_{\mathbf{m}g} \left(\gamma_i + 2\mathbf{T}_i^2 L_i(E_{\mathbf{m}}) \right) + \bar{\delta}_{\mathbf{m}g} \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) \right] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \\
 & - \sum_{i \in \mathcal{H}} [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \log\left(\frac{\eta_{i\mathbf{m}}}{4(1-\eta_{i\mathbf{m}})}\right) \Delta^{(\mathbf{m})} \left[\gamma_{\mathbf{m}} + 2\mathbf{T}_q^2 L_{\mathbf{m}} \bar{\delta}_{\mathbf{m}g} \right] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \\
 & + \left\langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{RV,fin}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \left. \right\} + \sum_n \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \Delta^{(\mathbf{m})} \left[\gamma_g^{\perp,r} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m}_g)] \right. \right. \\
 & \left. \left. + \gamma_g^{\perp} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) F_{\text{LM},\mu\nu}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m}_g)] \right] \right\rangle.
 \end{aligned}$$

- 1) terms proportional to perpendicular directions are null, as they arise from spin correlations that are absent in this case. The last two contributions can be dropped without further ado.
- 2) In the first and second contribution $E_{\mathbf{m}}$ has to be replaced with $E_{\max}$, as we don't have energy ordering.
- 3) care has to be devoted to the log -> we need to consider the matrix elements that are actually present (extra gluon, extra photon)
- 4) the third line has to be set to zero, as the collinearity $\mathbf{m}||\mathbf{n}$ is not singular in this case (look at the sector functions)

ABELIANISING THE GENERAL QCD RESULT

At **NNLO** things get more complicated:

1. The general results for NNLO QCD is based on energy ordering and sum over channels
2. In some cases the ordering is easy to reverse, thanks to the dependence over $E_{\max}$, in some cases it requires more care
3. Ongoing test on the agreement between the abelianised version of the fully general result for NNLO QCD and the “old fashion” calculation for mixed QCD×EW.
4. Fully general final formulas for mixed QCD×EW subtraction are within sight

...work in progress...

```
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% /. {Qe -> -(Qq + Qqbp)} // PowerExpand;
% /. {FVVqcdewfin[q, qbp, nu, l] -> 0};
% // Simplify // FunctionExpand // PowerExpand;
Collect[%, {_FLM, CF, Qq^2, Qqbp^2, Log[mu], Log[E1], Log[E2], Log[2], Log[1-z], pi^2}, FullSimplify]

CF Qq^2 FLM[q z, qbp, nu, l, z] ( - 8 (1+z^2) Log[1-z]^2 Log[z] / (-1+z) + pi^2 (11+(-6+z) z + 4 (1+z^2) Log[z]) / (3 (-1+z)) +
Log[1-z] ( 4 pi^2 (1+z^2) / (3 (-1+z)) + 4 (-1+z) (-8+7 z) - 6 (-1+z)^2 Log[z] + (1+z^2) Log[z]^2 - 8 (1+z^2) PolyLog[2, z] ) / (-1+z) +
Log[2] ( 4 pi^2 (1+z^2) / (3 (-1+z)) - 8 (1+z^2) Log[1-z] Log[z] / (-1+z) + 4 ((-1+z) (-8+7 z) + Log[z] (5-2 z^2 + (1+z^2) Log[z]) - 2 (1+z^2) PolyLog[2, z]) / (-1+z) ) +
Log[E1] ( 4 pi^2 (1+z^2) / (3 (-1+z)) - 8 (1+z^2) Log[1-z] Log[z] / (-1+z) + 4 ((-1+z) (-8+7 z) + Log[z] (5-2 z^2 + (1+z^2) Log[z]) - 2 (1+z^2) PolyLog[2, z]) / (-1+z) ) +
Log[mu] ( pi^2 (4+4 z^2) / (3-3 z) + 8 (1+z^2) Log[1-z] Log[z] / (-1+z) + -4 ((-1+z) (-8+7 z) + Log[z] (5-2 z^2 + (1+z^2) Log[z])) / (-1+z) + 8 (1+z^2) PolyLog[2, z] ) +
1 / (6 (-1+z)) (-6+6 z (-6+7 z) + Log[z] (132-6 z (19+13 z) + 75 Log[z] - 12 z (3+z) Log[z] + 14 (1+z^2) Log[z]^2 + 48 (-1+z^2) Log[1+z]) -
12 (9+3 (-2+z) z + (1+z^2) Log[z]) PolyLog[2, z] + 24 (-1+z^2 + (1+z^2) Log[z]) PolyLog[2, z^2] -
72 (1+z^2) PolyLog[3, 1-z] - 12 (1+z^2) PolyLog[3, z] - 24 (1+z^2) PolyLog[3, z^2] + 36 (1+z^2) Zeta[3]) ) + (1 <-> 2)
```

MAIN MESSAGES

- ▶ Mixed QCD×EW corrections to the Drell-Yan process are important to increase our control on SM parameters and search for new physics
- ▶ Theoretical calculations are reaching the required level of sophistication to provide prediction at sub-percent accuracy. Crucial role played by developments in QCD beyond NLO → importance of cross-checks among different methods
- ▶ Flexibility of existing analytical tools fundamental in avoiding a 'from-scratch' approach → previous studies of NCDY provides a robust framework and most of the required ingredients for subtraction
- ▶ Flexibility of existing numerical tools (and foreseen generalisations) fundamental to obtain reliable results in limited amount of time
- ▶ Generalising QCD×EW finite remainder [from subtraction procedure] to arbitrary processes should be possible by abelianising fully general QCD results

Theoretical developments

- ❖ Renormalisation [Dittmaier, Schmidt, Schwarz '20]
- ❖ Progress on two-loop master integrals [Bonciani, Di Vita, Mastrolia, Schubert '16][Heller, von Manteuffel, Schabinger '19][Hasan, Schubert '20]
- ❖ 2-loop amplitudes for $2 \rightarrow 2$ NCDY for massless leptons [Heller, von Manteuffel, Schabinger, Spiesberger '20]
- ❖ 2-loop amplitudes for $2 \rightarrow 2$ NCDY for massive leptons (retaining logarithms of lepton mass) [Armadillo, Bonciani, Devoto, Rana, Vicini '22]
- ❖ 2-loop amplitudes for $2 \rightarrow 2$ CCDY for massive leptons (retaining logarithms of lepton mass) [Armadillo, Bonciani, Devoto, Rana, Vicini '24]

On-shell Z/W production ($2 \rightarrow 1$ process)

- ❖ Analytic mixed QCD-QED corrections to the inclusive production of an on-shell Z [de Florian, Der, Fabre '18]
- ❖ Fully differential mixed QCD-QED corrections to the production of an on-shell Z [Delto, Jaquier, Melnikov, Rötsch '19]
- ❖ Total on-shell Z production cross section in fully analytic with mixed QCD-EW corrections [Bonciani, Buccioni, Rana, Vicini '20]
- ❖ Fully differential on-shell Z and W production with mixed QCD-EW corrections [Buccioni et al. '20] [Behring et al. '20]

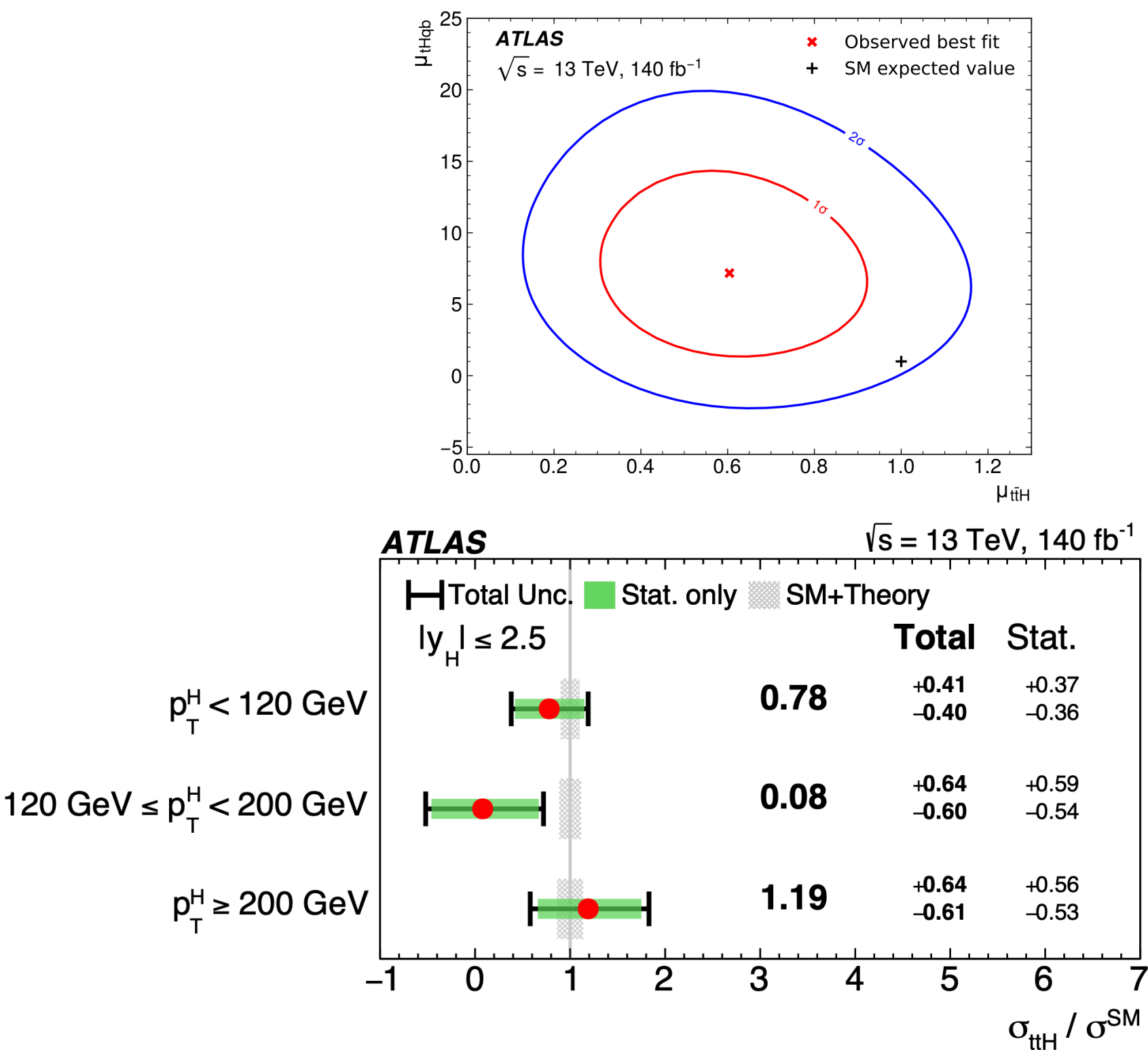
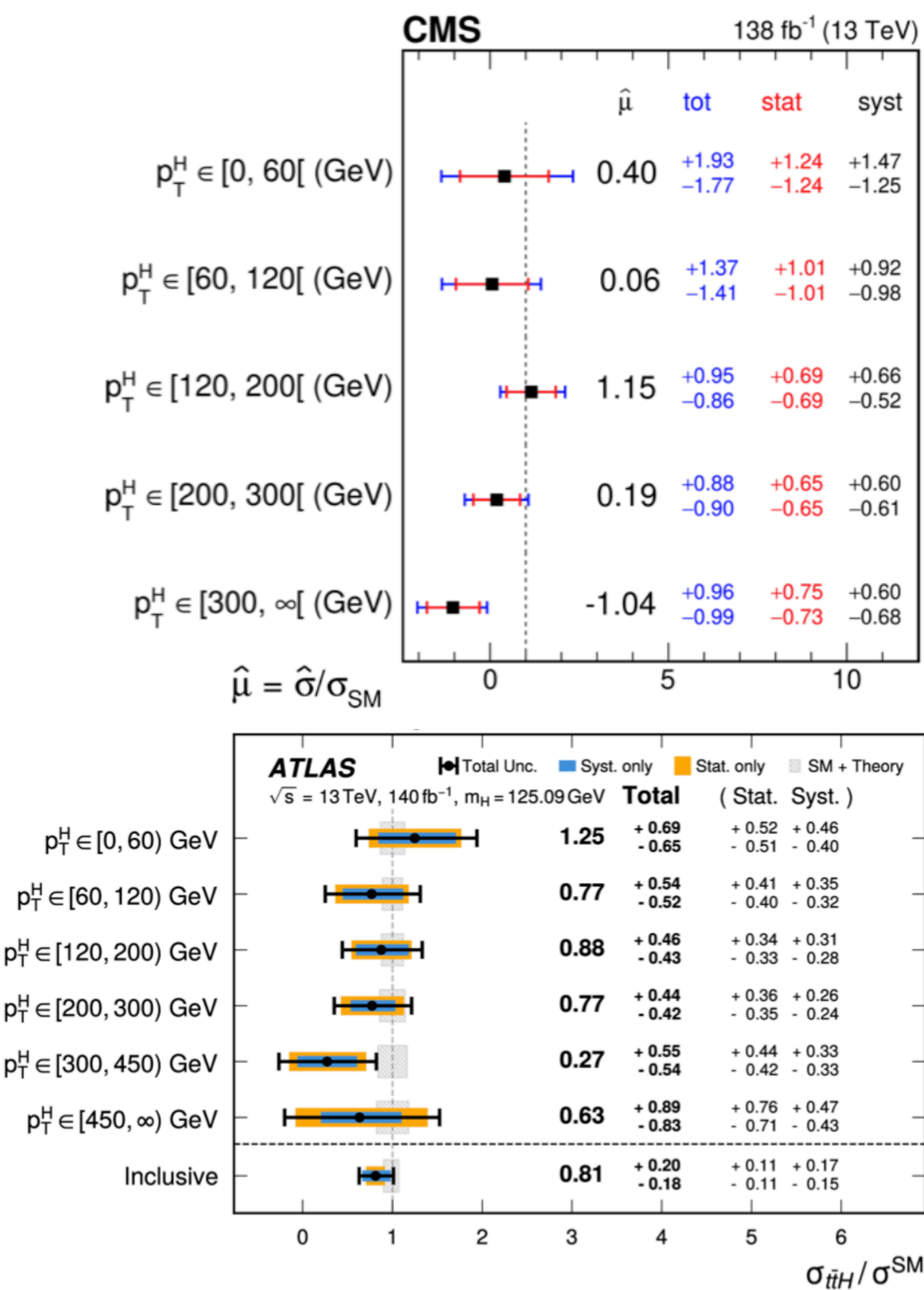
On-shell Z/W production ($2 \rightarrow 1$ process)

- ❖ Dominant mixed QCD-EW correction in the pole approximation NC and CC DY process [Dittmaier, Huss, Schwinn '14, 15]
- ❖ Approximate corrections available in parton showers based on factorised approach [Balossini et al. '10][Bernaciak, Wackerroth '12] [Barzè et al. '12, '13] [Calame et al. '17]
- ❖ Neutrino-pair production including mixed QCD-QED corrections [Cieri, de Florian, Der, Mazzitelli '21]
- ❖ Neutrino-pair production including mixed QCD-EW corrections [Buonocore, Grazzini, Kallweit, Savoini, Tramontano '21]
- ❖ Complete mixed QCD-EW corrections to NC and CC DY process [Bonciani, Buonocore, Grazzini, Kallweit, Rana, Tramontano, Vicini '22]

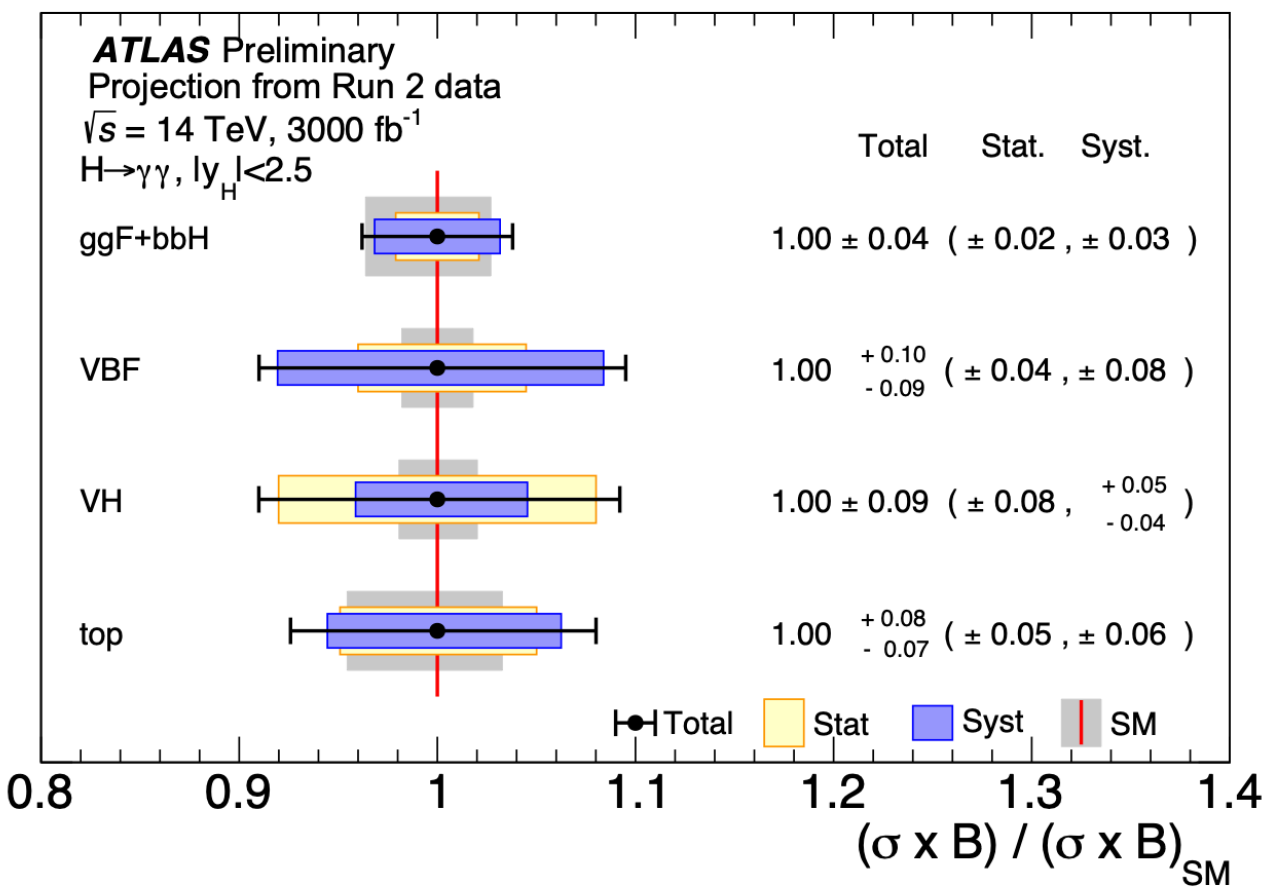
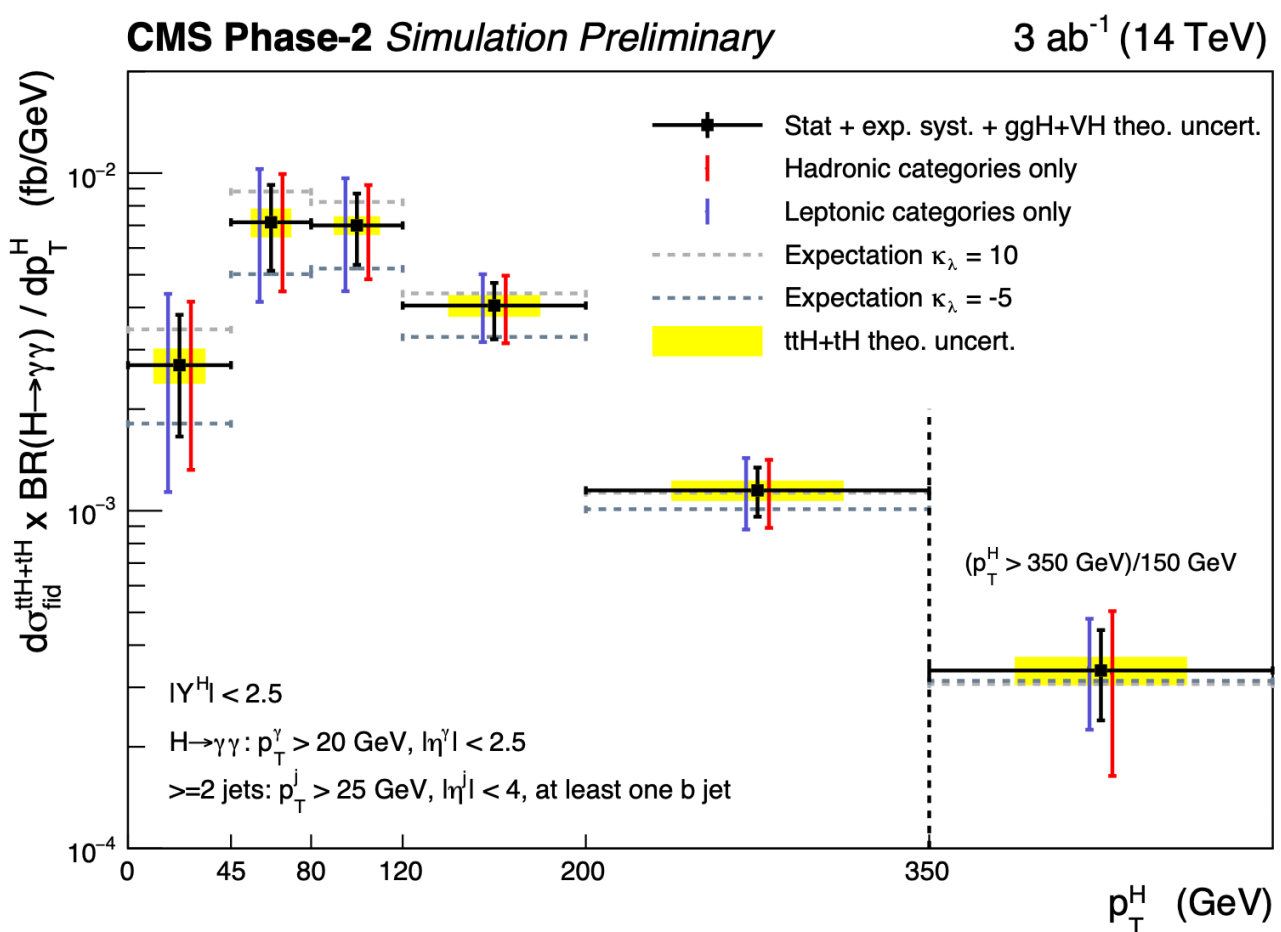
HIGGS PRODUCTION IN ASSOCIATION TO A TOP-QUARK PAIR

MAIN MESSAGES

- ▶ $t\bar{t}H$ crucial for precise determination of Higgs-top Yukawa, Higgs properties, top-quark precision measurements, new physics searches.

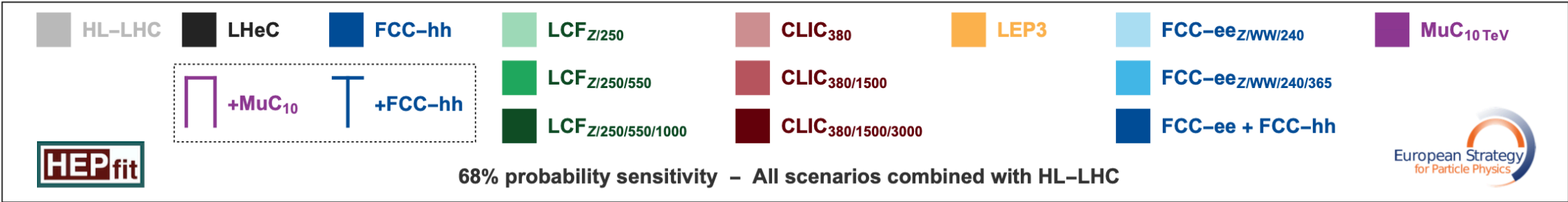


First differential ATLAS and CMS measurements
[2407.10896, 2407.10904]
Present status [2510.23755]
Projections for the future [1902.00134]

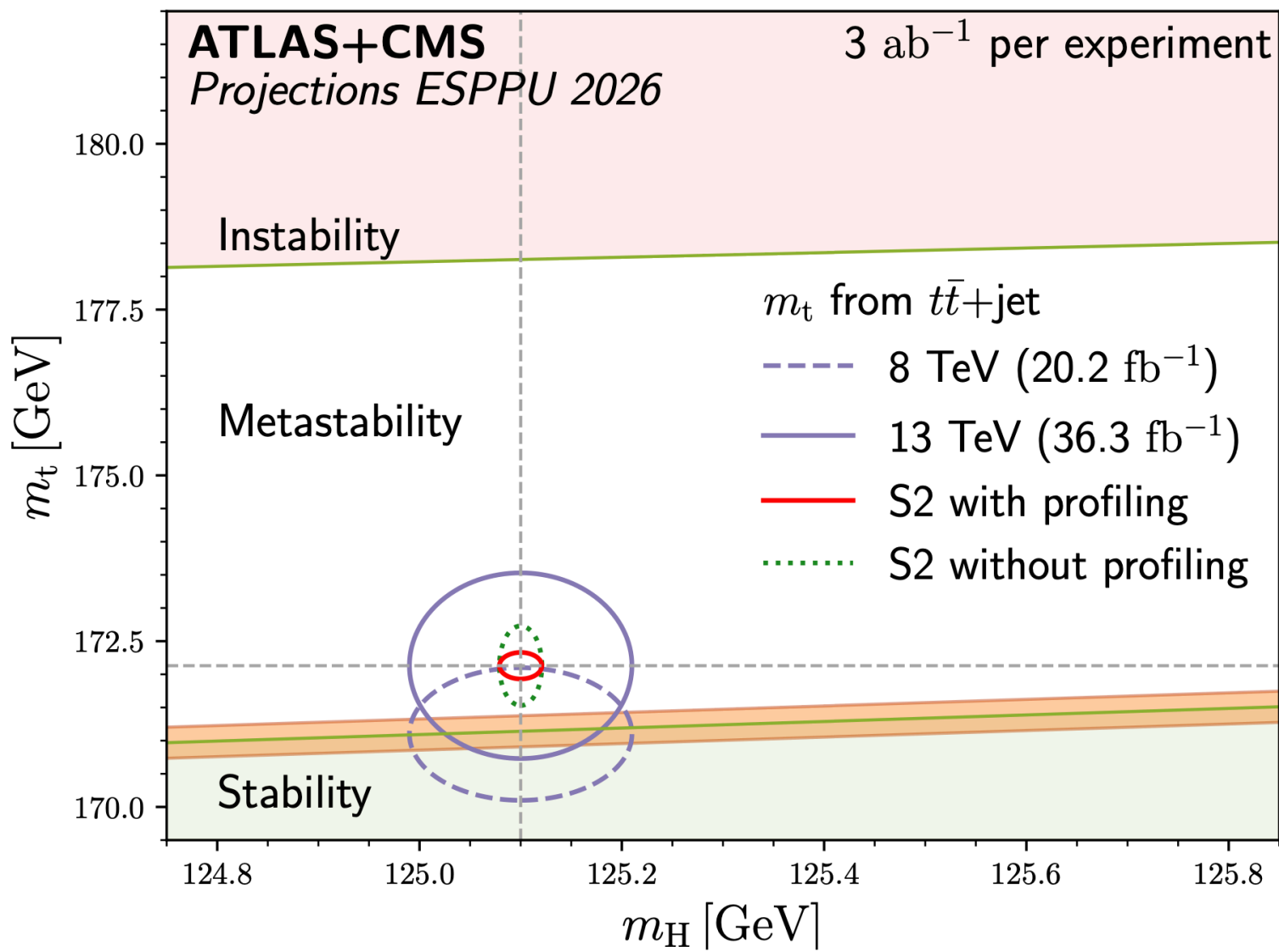
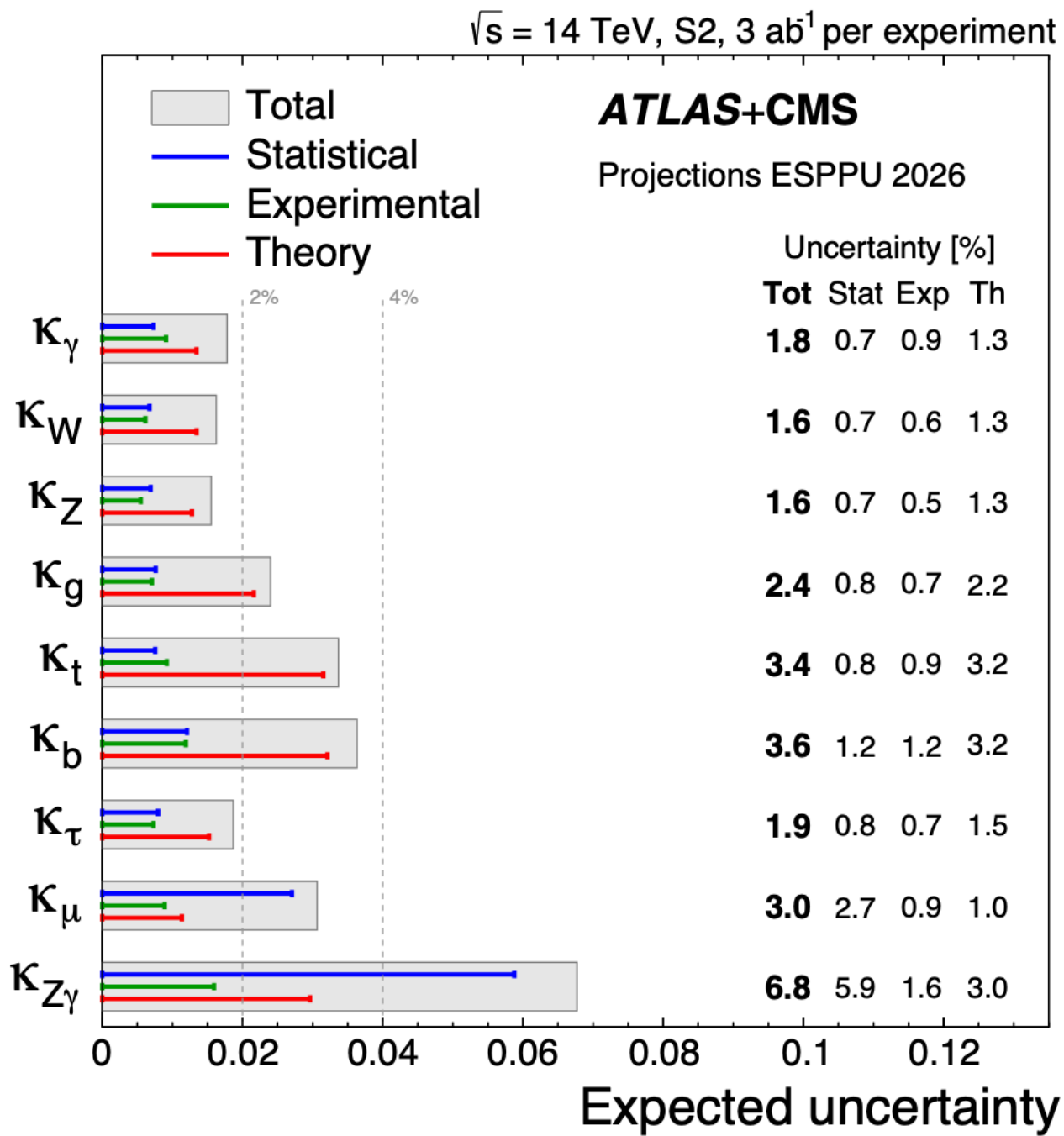
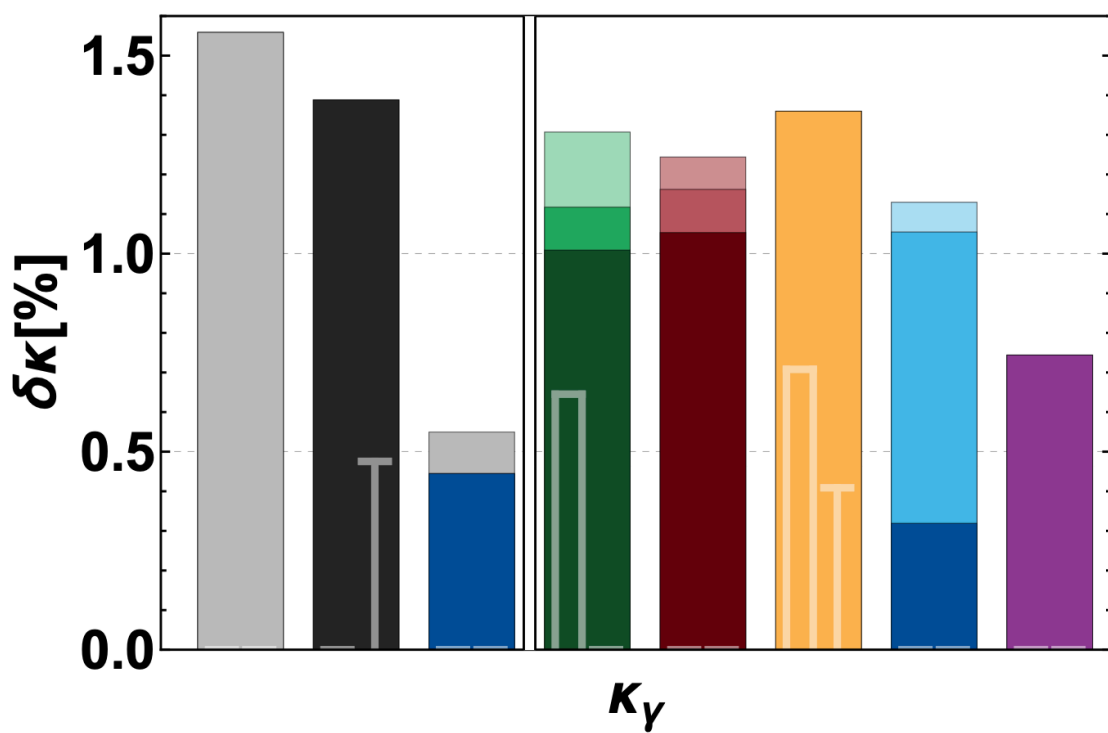
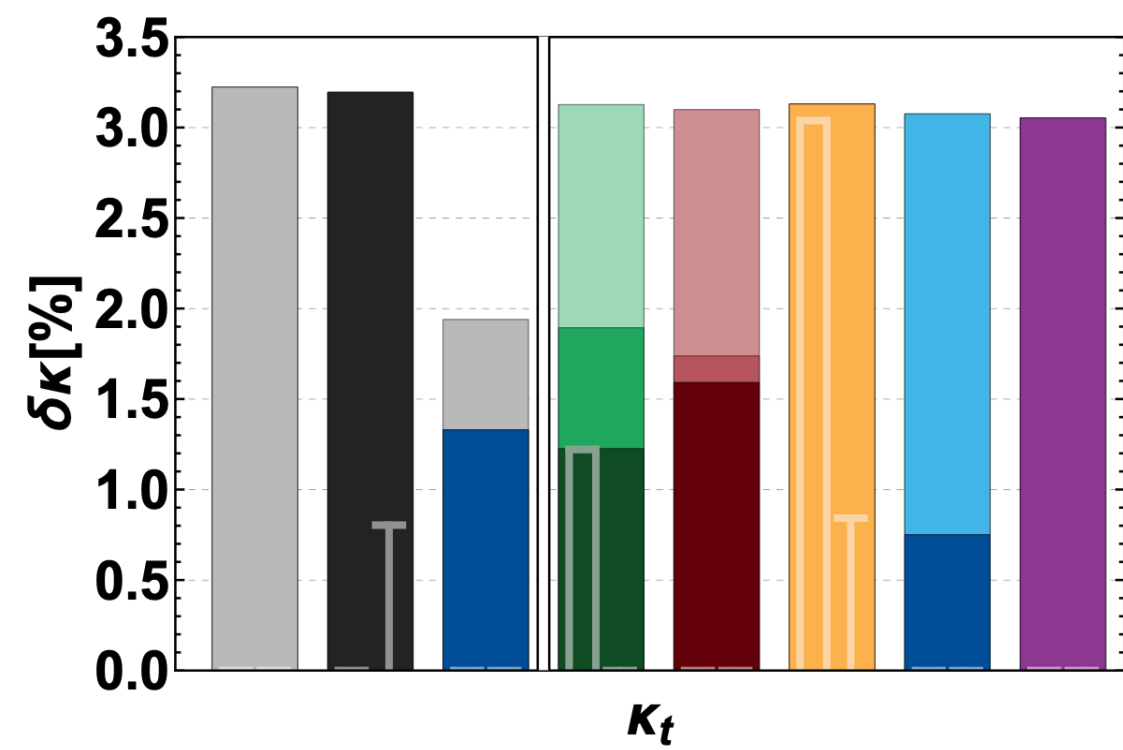


MAIN MESSAGES

- ▶ $t\bar{t}H$ crucial for precise determination of Higgs-top Yukawa, Higgs properties, top-quark precision measurements, new physics searches.



$t\bar{t}H$ part of an extremely rich research program at the core of the scientific strategy for the future!



MAIN MESSAGES

- ▶ $t\bar{t}H$ crucial for precise determination of Higgs-top Yukawa, Higgs properties, top-quark precision measurements, new physics searches.
- ▶ The present (and future) level of precision calls for increasingly complicated calculations, which span different energy scales

PRECISION AND REALISM

Parton shower PS_{N^yLL}
and hadronisation

☑ N^yLL resummation

Hard Process N^xLO

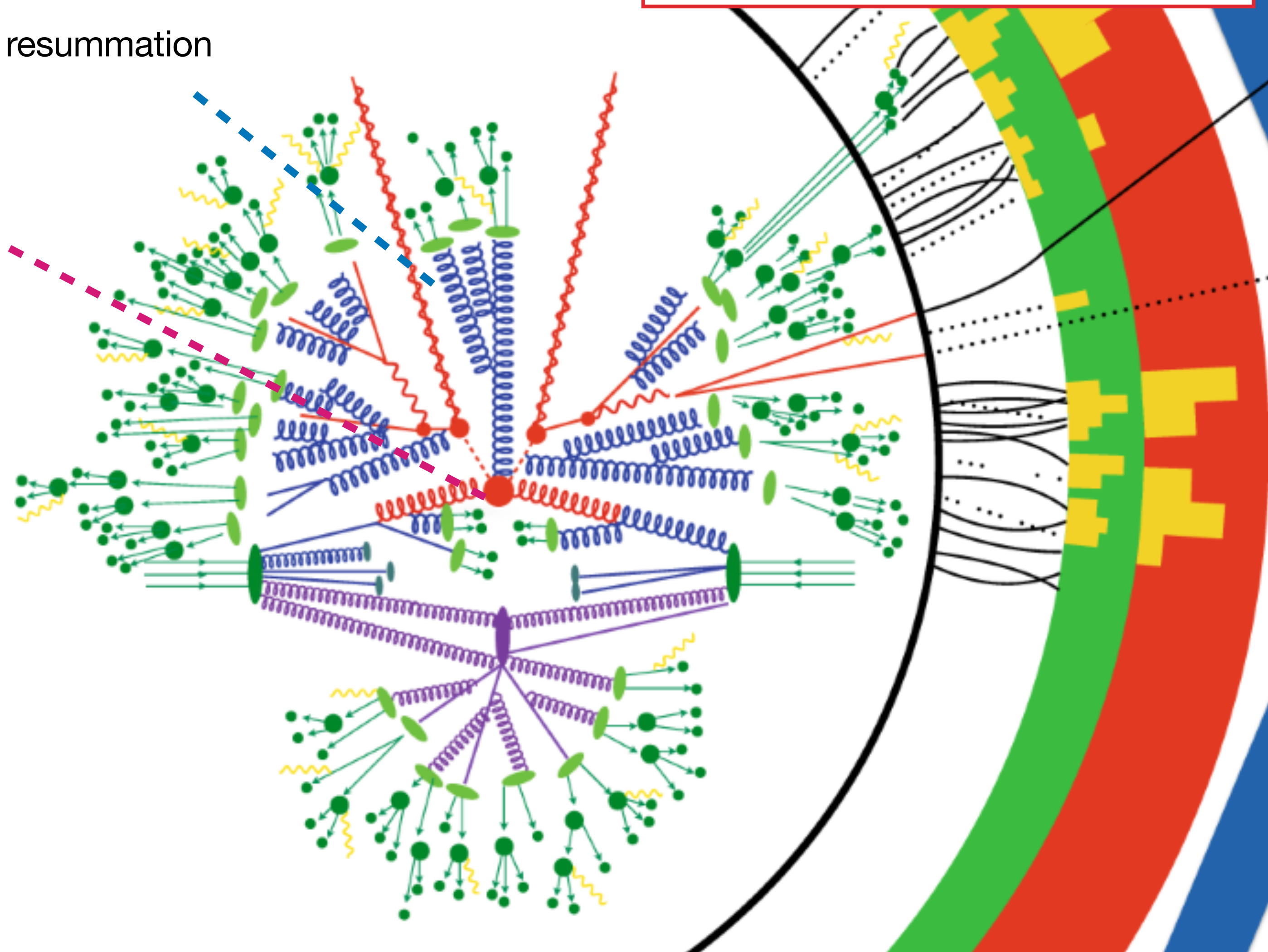
☑ High precision

Matching
 $N^xLO + PS_{N^yLL}$

☑ Realistic simulation of LHC events

State-of-the-art

- ▶ Fixed-order $NNLO_{QCD}$ with approximated 2loop
(Catani), Devoto et al. [2210.07846, 2411.15340]
- ▶ NLO_{EW} and NLO_{QCD}
Frixione et al. [1504.03446, 1804.10017]
- ▶ Matching with soft-gluon NNLL resummation
Kulesza et al. [1509.02780, 1704.03363, 2001.03031]
Broggio et al. [1510.01914, 1611.00049, 1907.04343]
LHCHWG Report [2503.15043]
- ▶ $NLO_{QCD} + PS_{LL}$
Frederix et al. [1104.5613]
Hartanto et al. [1501.04498]



MAIN MESSAGES

- ▶ $t\bar{t}H$ crucial for precise determination of Higgs-top Yukawa, Higgs properties, top-quark precision measurements, new physics searches.
- ▶ The present (and future) level of precision calls for increasingly complicated calculations, which span different energy scales
- ▶ Approximations/estimations/incremental improvements can guide the progression of the theoretical predictions

MINNLOPS

[Christian Biello | Rencontres de Moriond 2026]

Here $PS=PS_{LL}$

POWHEG $\times+j$ @NLOPS

- ✓ NLOPS accuracy for $\times+j$
- ✓ Natural matching with the first additional shower radiation

Monni, Nason, Re, Wieseemann,
Zanderighi [1908.06987]

MiNLO' framework

- ✓ NLO accuracy for $\times$ from $\times+j$ generator

Hamilton, Nason, Oleari,
Zanderighi [1212.4504]

Transverse momentum resummation

(Bozzi), Catani, (de Florian), Grazzini
[0508068, 1011.3918]
(Bizon), Monni, Re, (Rottoli), Torrielli
[1604.02191, 1705.09127]

MiNNLOPS

- ✓ NNLO accuracy for $\times$ production + PS
- ✓ Applied to several processes with different features

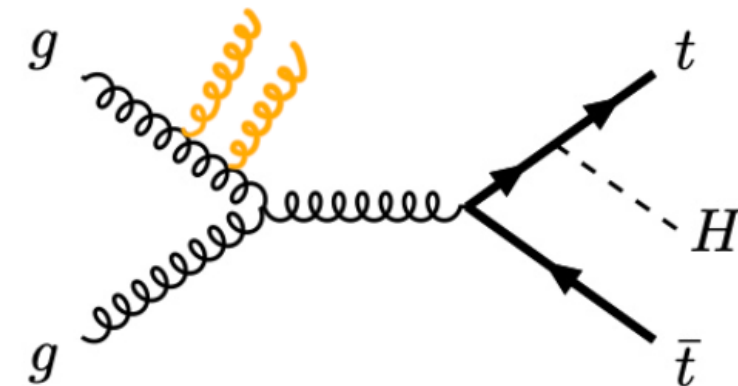
The method

{ PROCESS, CORRECTION }

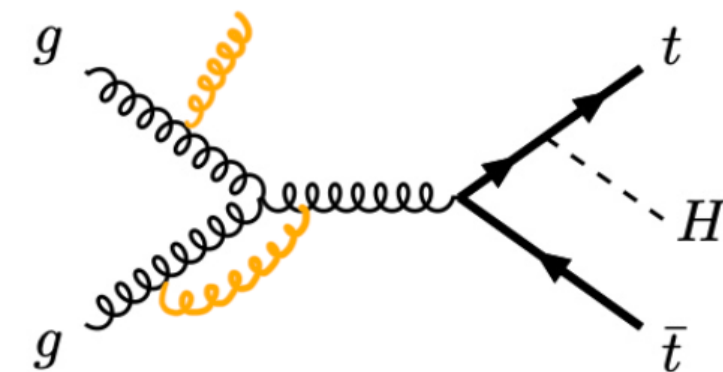
Do I **HAVE** all the “exact” ingredients?

Hard Process N^2LO

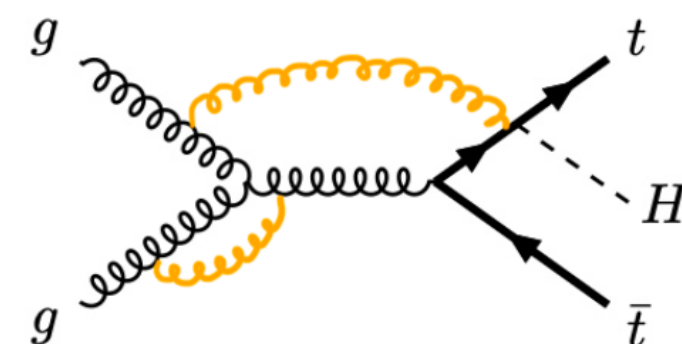
▶ Double-real
[OpenLoops]



▶ Real-virtual
[OpenLoops]



▶ Double virtual



Matching $N^2LO + PS_{LL}$

▶ NLO+PS accuracy $t\bar{t}H + j$ [POWHEG] 

▶ NLO+PS($t\bar{t}H$)&NLO+PS($t\bar{t}H + j$) [MiNLO'] 

▶ Transverse momentum resummation 

Monni, Nason, Re,
Wiesemann, Zanderighi
[1908.06987]

MiNNLOps

$X = t\bar{t}H$

☒ NNLO accuracy for
x production + PS

☒ Applied to several
processes with
different features

*references in the backup

{ PROCESS, CORRECTION }

YESDo I **WANT** all the “exact” ingredients?**NO**

need a workaround

Wait for the ingredients to become
(publicly) available/compute them myself
[might not match the “urgency” for results]

Develop a controlled approximation strategy

- ▶ Construct the best proxy for the missing bits
- ▶ Assign conservative uncertainties
- ▶ Ensure consistency with known constraints

Thorough validation

- ▶ Directly usable within pheno. frameworks
- ▶ Captures relevant physics at required precision

Reliable result with quantified confidence

*

* Exact results are crucial for “final” checks and should be incorporated as soon as they become available

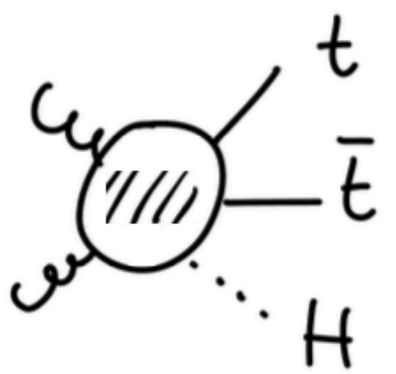
DIFFERENT “LIMITS”

[Christian Biello | Rencontres de Moriond 2026]

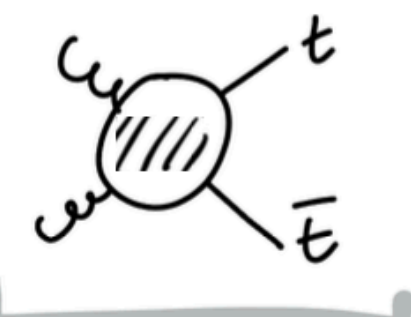
Soft Higgs-boson approximation

based on Kniehl, Spira [hep-ph/9505225] and refs. therein

$$\lim_{q \rightarrow 0} \text{A}^{\text{bare}} \xrightarrow{q} \sim \underbrace{\mathcal{J}^{(0)}(q)}_{\frac{m}{v} \left(\sum_i \frac{m}{q \cdot p_i} \right)} \text{A}^{\text{bare}}$$

 $\sim \mathcal{F}(\alpha_s(\mu^2), \mu/m) \mathcal{J}^{(0)}(p_H)$

Soft limit
of the heavy-quark
scalar form factor



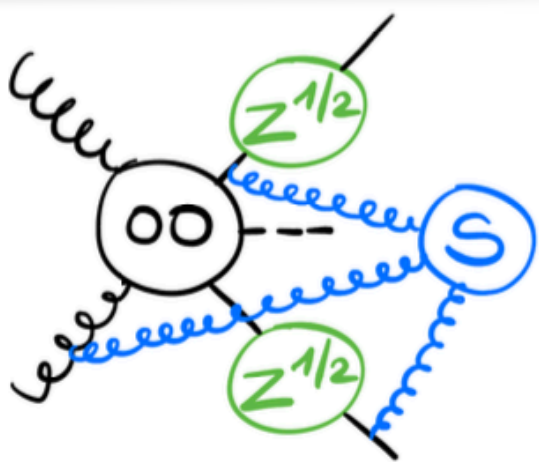
from grids

First application in $t\bar{t}H$:
Catani, Devoto, Grazzini, Kallweit,
Mazzitelli, Savoini [2210.07846]

Two-loop $t\bar{t}$ grids:
Bärnreuther, Czakon, Fiedler
[1312.6279]

Massification of top-quarks

based on Mitov, Moch [hep-ph/0612149]
and Wang, Xia, Yang, Ye [2312.12242]



$$\text{Diagram} = \left[\mathcal{F}^{(2)}\left(\log\left(\frac{m_t}{\mu}\right)\right) + \mathcal{S}^{(2)}\left(\frac{m_t}{\mu}, \{p\}\right) \right] \text{Diagram}_{m_t=0}$$
$$+ \mathcal{F}^{(1)}\left(\log\left(\frac{m_t}{\mu}\right)\right) \text{Diagram}_{m_t=0} + \text{Diagram}_{m_t=0} + \mathcal{O}\left(\frac{m_t}{\mu}\right)$$

2→3 two loop
with massless tops

LC amps: Badger et al. [2107.14733]
LC C++ library: bbH MiNNLOps [2412.09510]
checked against Savoini’s QQHamp [2411.15340]
FC C++ library: Badger et al. [2412.06519]

Change of Yukawa scheme: $\overline{\text{MS}} \rightarrow \text{OS}$
Numerical optimisation with quadruple-precision fallback

The two approximations have been built in POWHEG with point-wise checks against Savoini’s FO code

COMBINED APPROXIMATION

*references in the backup

New approach in MiNNLOPS: point-wise combination (CA) of two well established approx.* as done for the differential NNLO calculation in MATRIX [2411.15340]

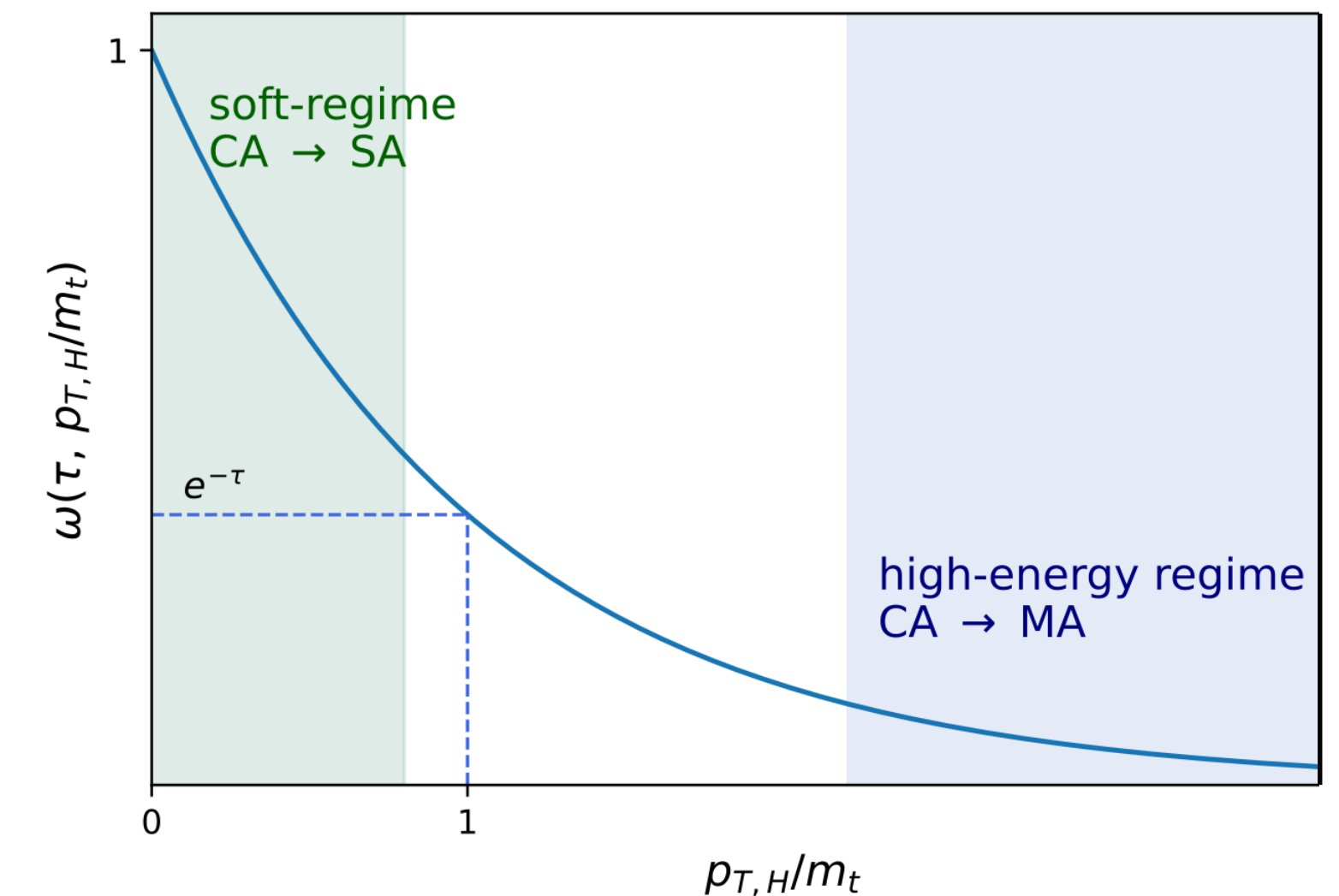
$$CA = \omega(\tau)SA + (1 - \omega(\tau))MA$$

Soft Higgs approximation (SA)

Massification of
top-quarks (MA)

Smooth weight function:

$$\omega(\tau, p_{T,H}/m_t) = \exp(-\tau p_{T,H}/m_t)$$



Conservative estimate of the uncertainties: $CA(\mu) = \omega(\tau) SA(\eta m_{t\bar{t}H}) + [1 - \omega(\tau)] MA(\eta m_{t\bar{t}H})$

- ▶ Variation of the scale η at which the combined approximation is applied
- ▶ Variation of τ entering ω
- ▶ Discrepancy $\delta = |CA^{(1)}/H^{(1)} - 1|$ between approx. and exact hard-virtual @1loop

Variation of η and τ independently around $\eta = \tau = 1 \rightarrow$ nine-point variation

Inclusion of $(1 \pm \delta) CA^{(2)}$ band at each phase-space point $\rightarrow$ two-point variation

} 11-point scale variation envelope

MAIN MESSAGES

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- ▶ The present (and future) level of precision calls for increasingly complicated calculations, which span different energy scales
- ▶ Approximations/estimations/incremental improvements can guide the progression of the theoretical predictions
- ▶ Flexible generators can be built on this principle, allowing for precise predictions (validation campaigns crucial to stress-test the combination and access meaningful uncertainties related to the procedure).

VALIDATION

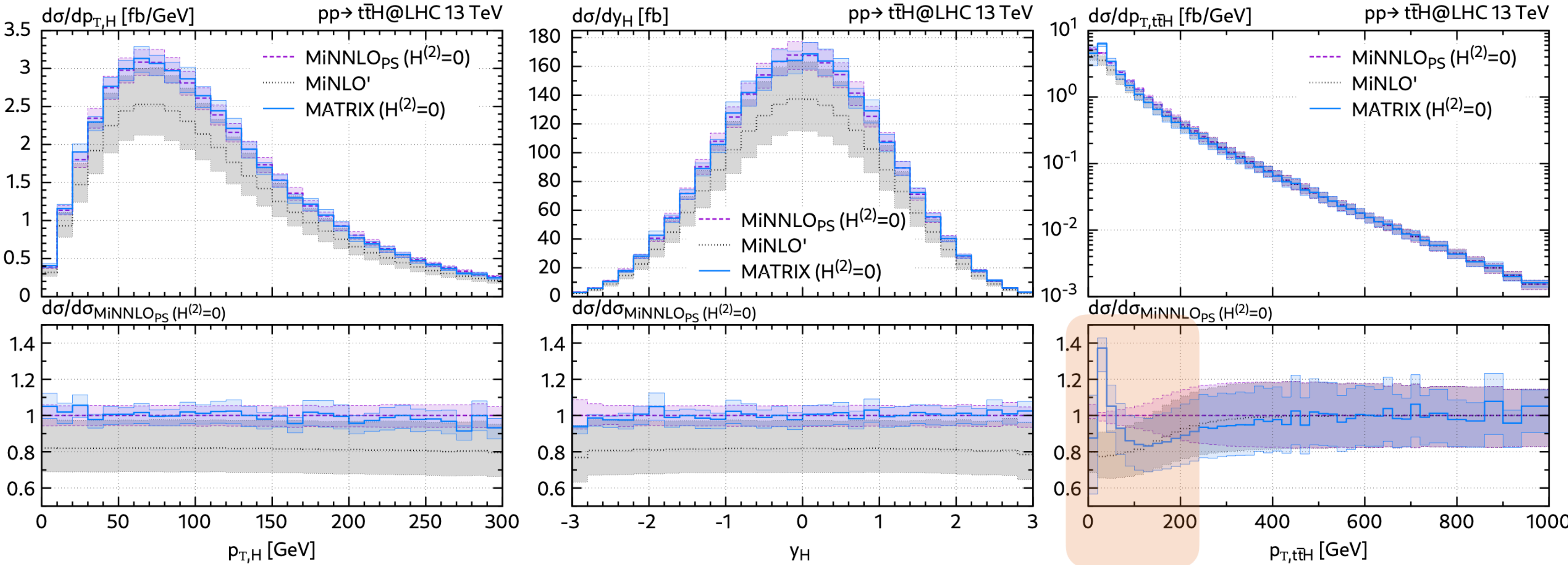
1. VALIDATION OF THE FIXED-ORDER COMPONENT

- ▶ MiNNLOPS achieves NNLO accuracy while consistently incorporating resummation and PS effects
- ▶ Benchmark against independent predictions from MATRIX (*Catani, Devoto, Grazzini, Kallweit, Mazzitelli, Savoini [2210.07846, 2411.15340]*)
- ▶ Double-virtual finite remainder set to zero

- ▶ NNLO-terms (MiNLO' $\rightarrow$ MiNNLOPS)
leads to $O(+20\%)$ on $\sigma_{\text{inclusive}}^{t\bar{t}H}$

	MINLO'	MINNLO _{PS} ($\mathbb{H}^{(2)} = 0$)	MATRIX ($\mathbb{H}^{(2)} = 0$)
$\sigma_{\text{inclusive}}^{t\bar{t}H}(\sqrt{s} = 13 \text{ TeV})$	$0.3864(2)^{+19.2\%}_{-16.2\%} \text{ pb}$	$0.4734(2)^{+5.4\%}_{-5.9\%} \text{ pb}$	$0.4742(9)^{+4.7\%}_{-6.0\%} \text{ pb}$

- ▶ Excellent agreement at differential level for inclusive observables
- ▶ If resummation effects become relevant ($p_{T,t\bar{t}H} \rightarrow 0$), PS matching essential



VALIDATION

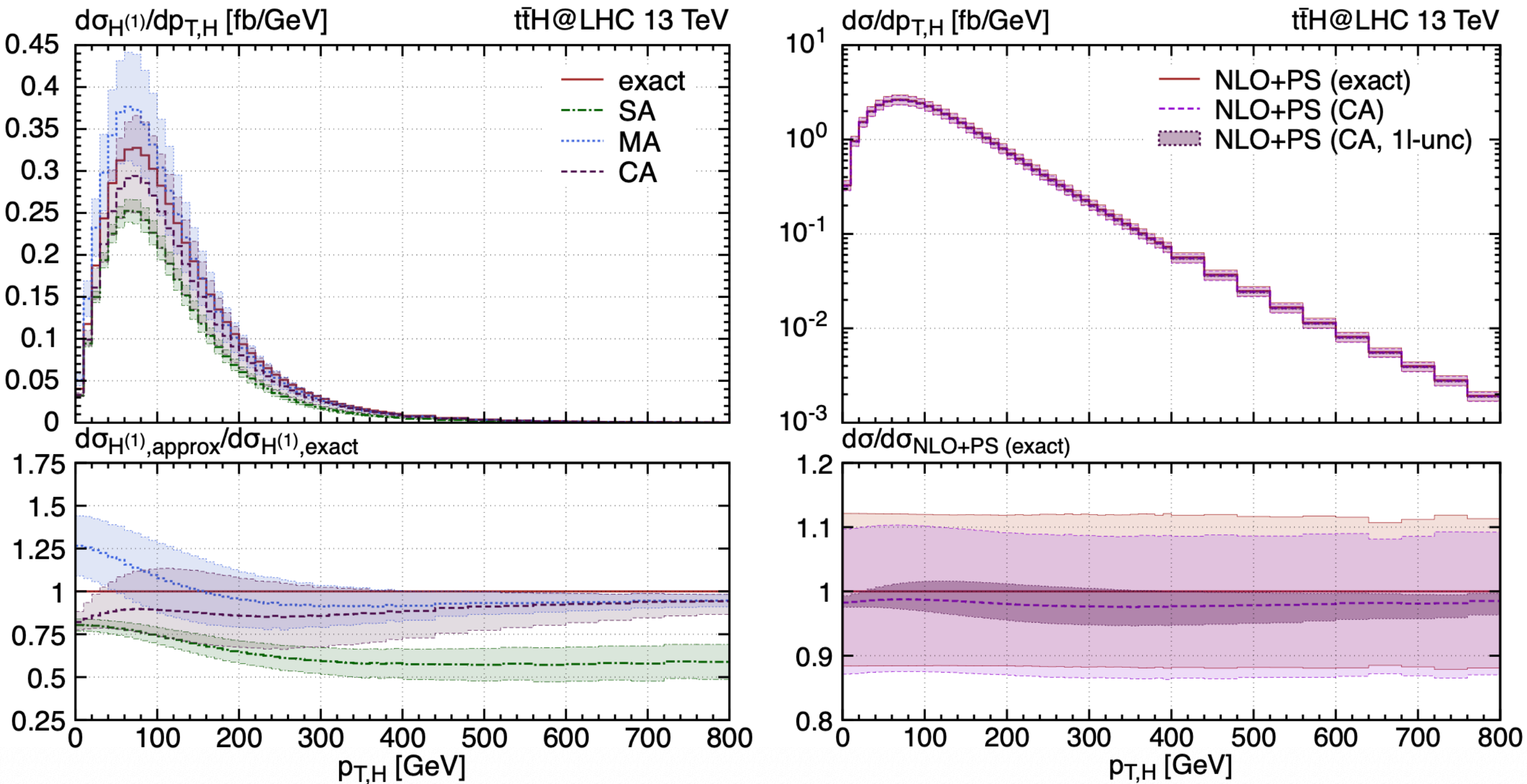
2. VALIDATION OF THE COMBINED APPROXIMATION@1LOOP

- ▶ Validation of the combined approximation at one-loop and NLO where the exact result is known.
- ▶ Inclusive xsection: 1-CA/exact ~ 1.5% well within systematic assigned to the approx.
- ▶ NLO ~ 1.6×LO: sizeable correction driven by dynamical scale choice ($\mu_R = \mu_F = m_{t\bar{t}H}$)

	LO+PS	NLO+PS (exact)	NLO+PS (CA)
$\sigma_{\text{inclusive}}^{t\bar{t}H}(\sqrt{s}=13\text{ TeV})$	$0.2572(1)^{+26.5\%}_{-19.6\%}\text{ pb}$	$0.4096(1)^{+12.0\%}_{-11.6\%}\text{ pb}$	$0.4035(1)^{+11.5\%[+2.6\%]}_{-11.3\%[-1.9\%]}\text{ pb}$

$$d\sigma_{\mathbb{H}^{(1)}} = \frac{\alpha_s(M)}{2\pi} \mathbb{H}^{(1)}(M) d\sigma_{\text{LO}}$$

- ▶ SA significantly underestimates the exact prediction @large $p_{T,H}$
- ▶ MA overestimates the exact prediction @small- $p_{T,H}$, while approaching the exact result at high- $p_{T,H}$



VALIDATION

3. PROJECTION @NNLO [NO AVAILABLE EXACT RESULTS]

► The calculation is exact but the genuine two-loop hard-virtual coefficient function

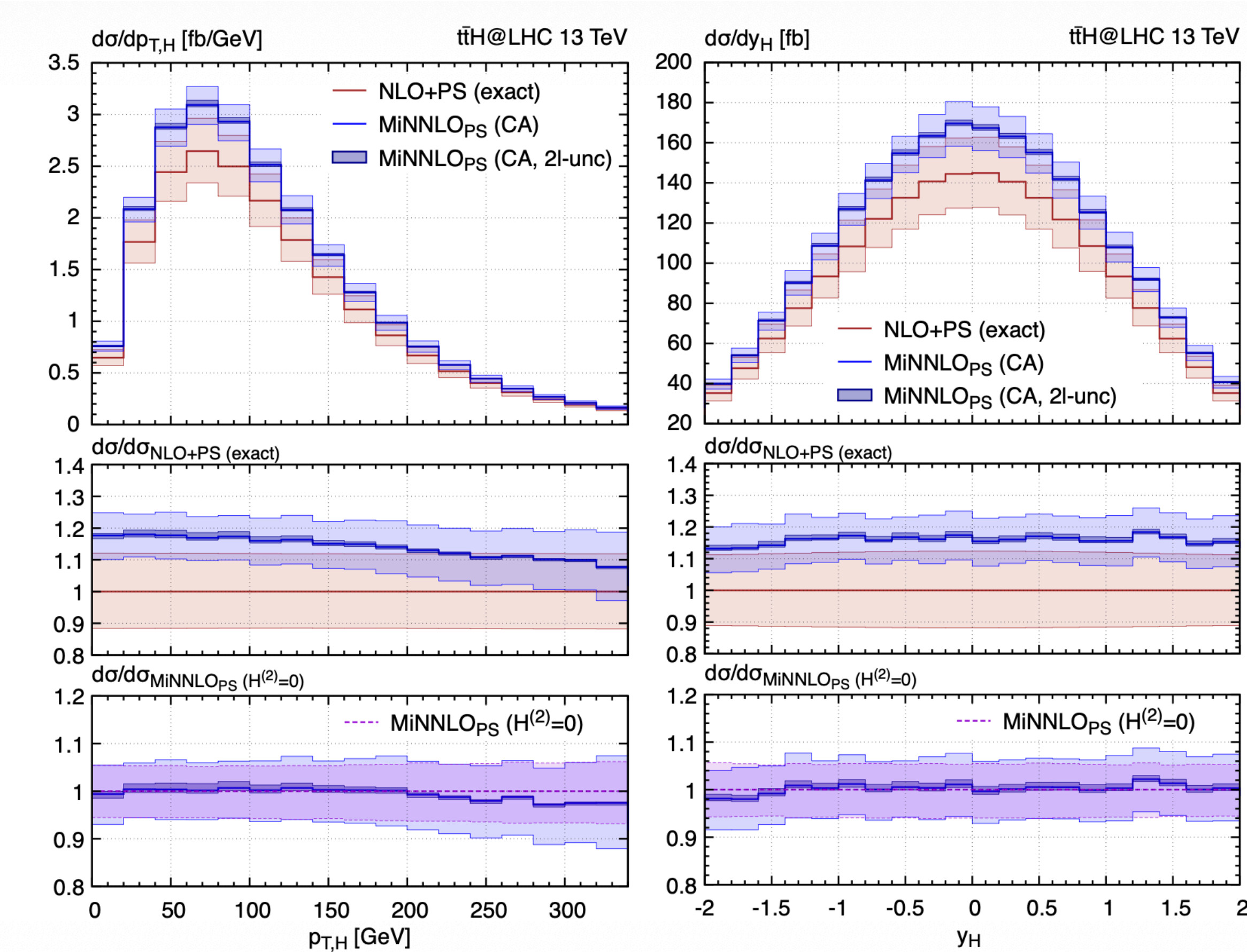
	NLO+PS (exact)	MINNLO _{PS} ($\mathbb{H}^{(2)} = 0$)	MINNLO _{PS} (CA)
$\sigma_{\text{inclusive}}^{t\bar{t}H}(\sqrt{s} = 13 \text{ TeV})$	$0.4096(1)^{+12.0\%}_{-11.6\%} \text{ pb}$	$0.4734(2)^{+5.4\%}_{-5.9\%} \text{ pb}$	$0.4778(5)^{+6.1\%[+1.04\%]}_{-6.3\%[-0.57\%]} \text{ pb}$

► NLO+PS vs MiNNLOPS ($H^2=0$): including even partial NNLO contributions makes a big difference

► Numerical impact of 2loop coefficient in the CA is generally small (third panel)

► Uncertainty related to the 2loop estimate is well within the scale uncertainty

► The combined approximation behaves as expected and no uncontrolled effects appear



MAIN MESSAGES

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- ▶ The present (and future) level of precision calls for increasingly complicated calculations, which span different energy scales
- ▶ Approximations/estimations/incremental improvements can guide the progression of the theoretical predictions
- ▶ Flexible generators can be built on this principle, allowing for precise predictions (validation campaigns crucial to access meaningful uncertainties related to the procedure).
- ▶ Studies of differential distributions are possible with a fully general and efficient generator (including off-shell effects in different fiducial phase spaces).

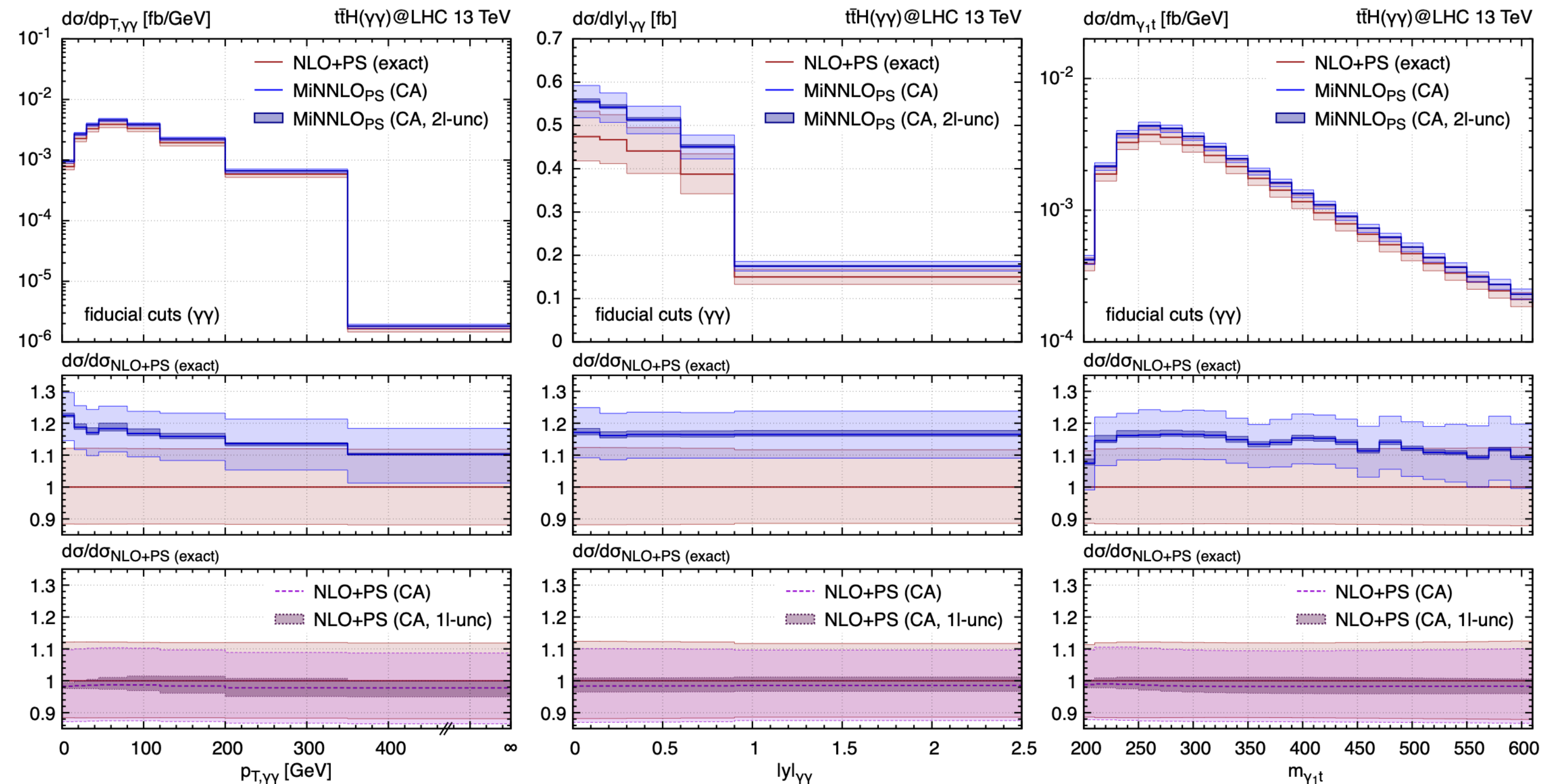
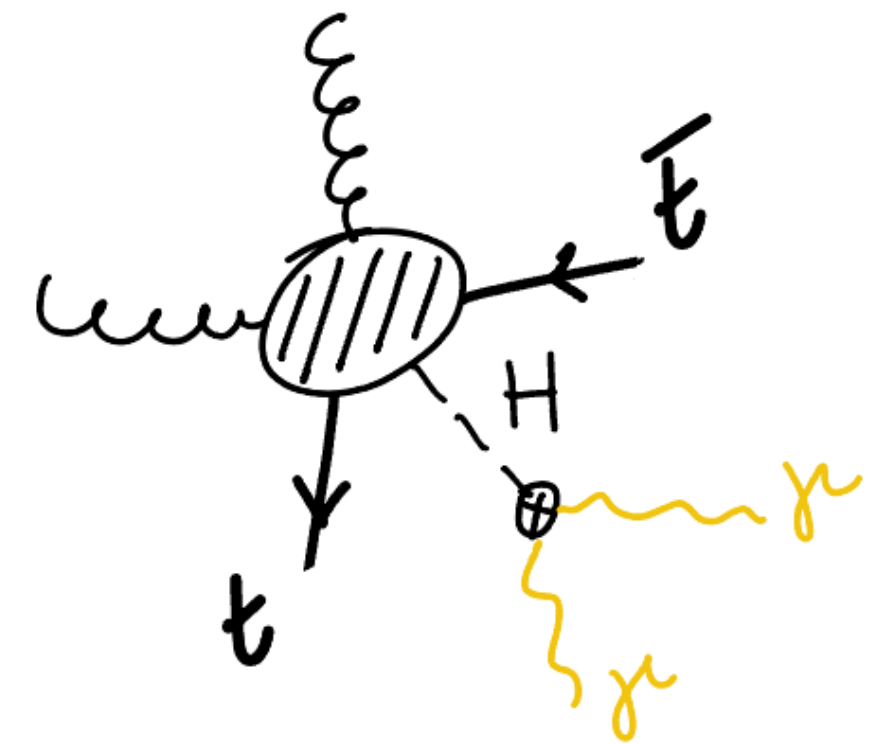
HIGGS-BOSON DECAY

- ▶ Showered LHE events with the option to decay the Higgs into photons via PYTHIA8
- ▶ fiducial region inspired by experimental analysis

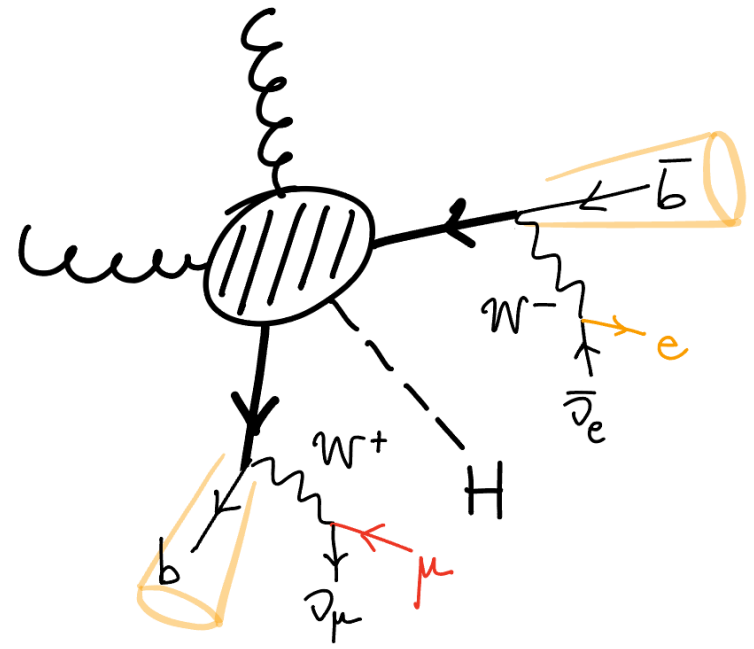
$$p_{T,\gamma_1}/m_{\gamma\gamma} > 1/3, \quad p_{T,\gamma_2}/m_{\gamma\gamma} > 1/4,$$

$$|\eta_{\gamma_{1,2}}| < 2.5,$$

- ▶ Non-flat positive corrections for Higgs transverse momentum and invariant mass of leading photon and top quark
- ▶ CA approach validated for these fiducial studies at NLO+PS



TOP-QUARK DECAY



- Tree-level spin-correlated off-shell top-quark effects in the generator

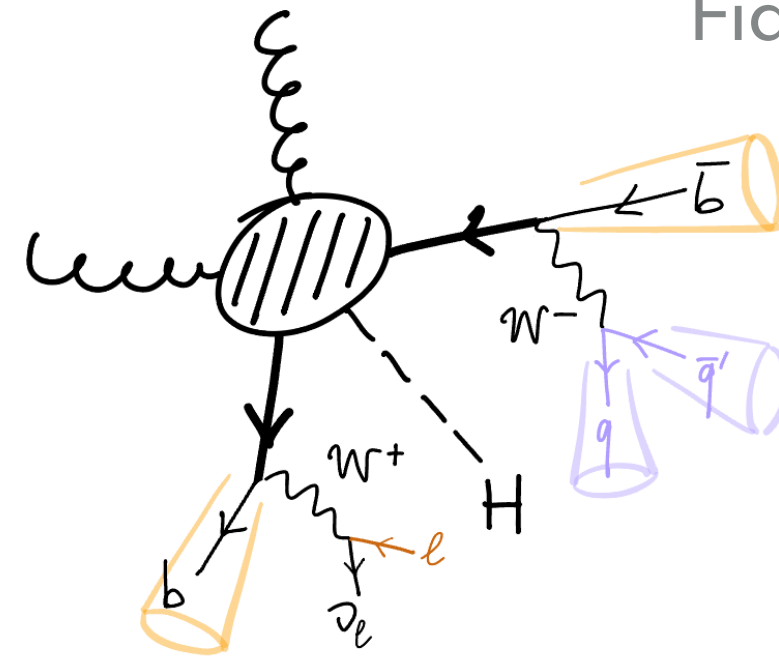
- Dilepton channel with fiducial cuts

$$p_{T,\ell} > 10 \text{ GeV}, |\eta_\ell| < 2.5$$

$$\max[p_{T,\ell}, \ell \in \{e, \mu\}] > 15 \text{ GeV}$$

$$p_{T,j} > 25 \text{ GeV}, |\eta_j| < 2.5$$

$$\sum_{\Delta R(i,\ell) < 0.4} p_{T,i} < 0.35 p_{T,\ell}$$



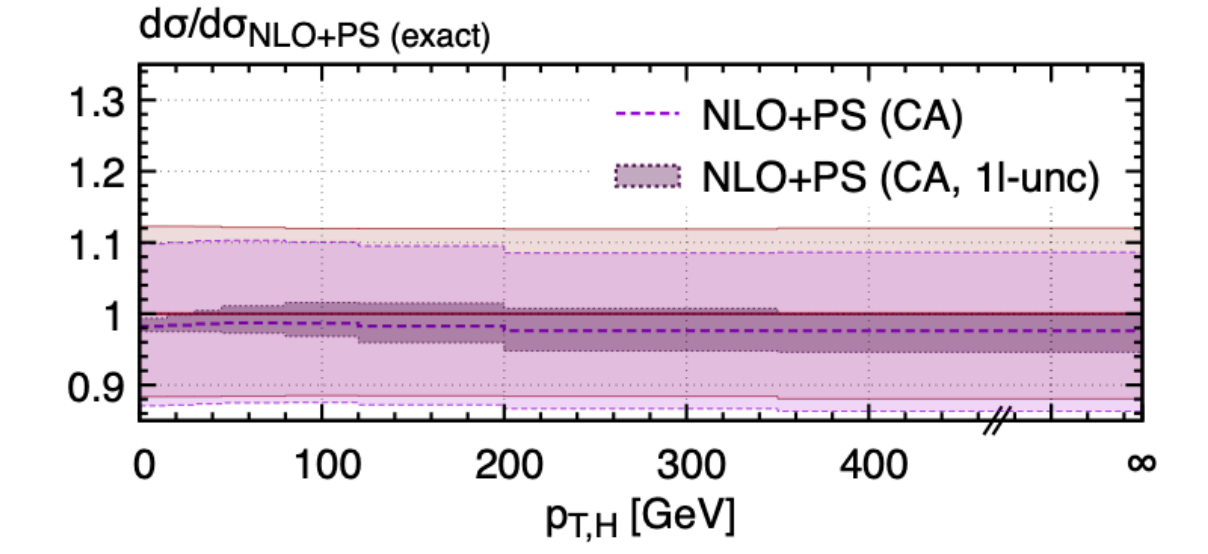
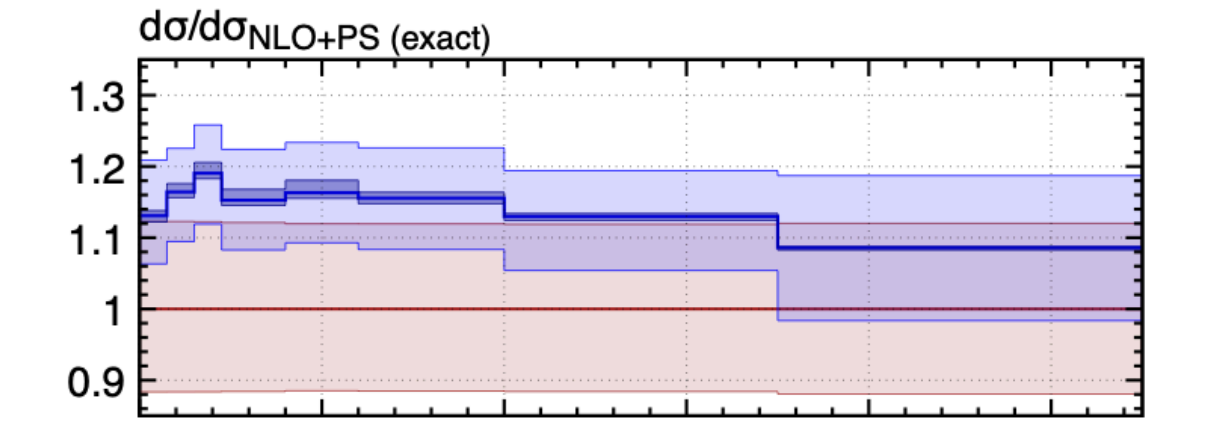
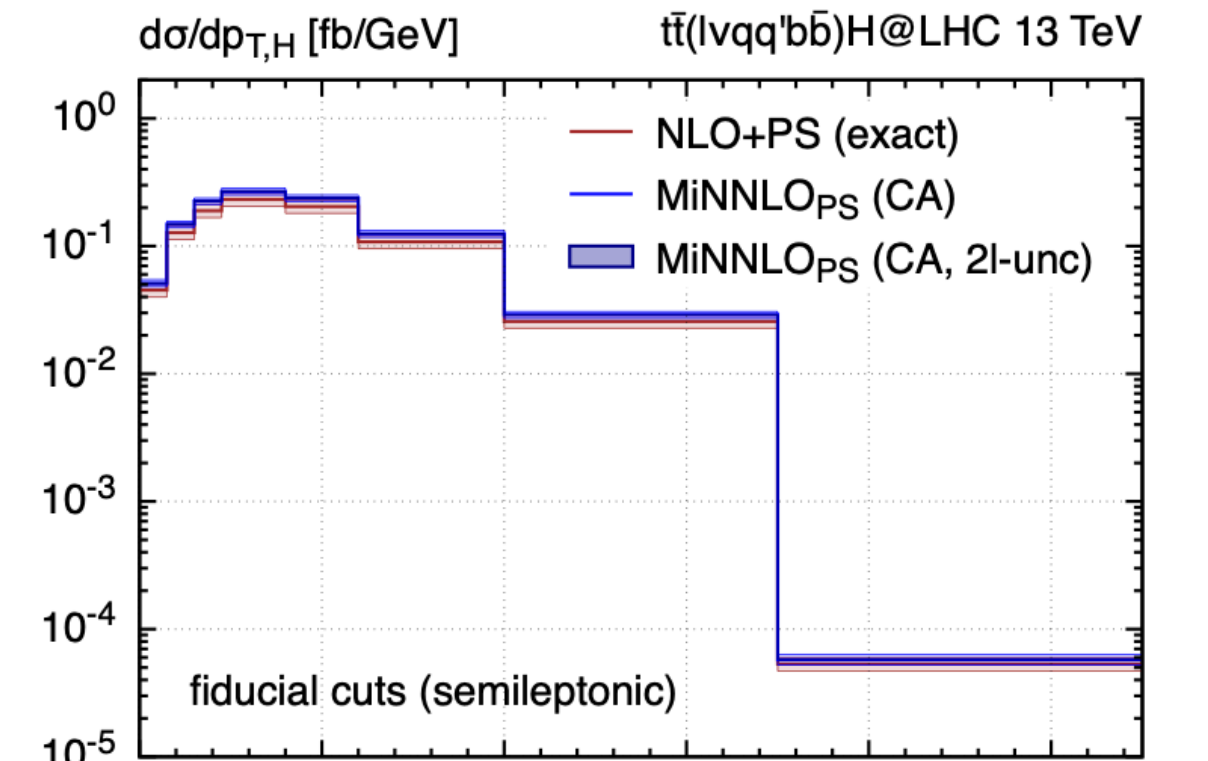
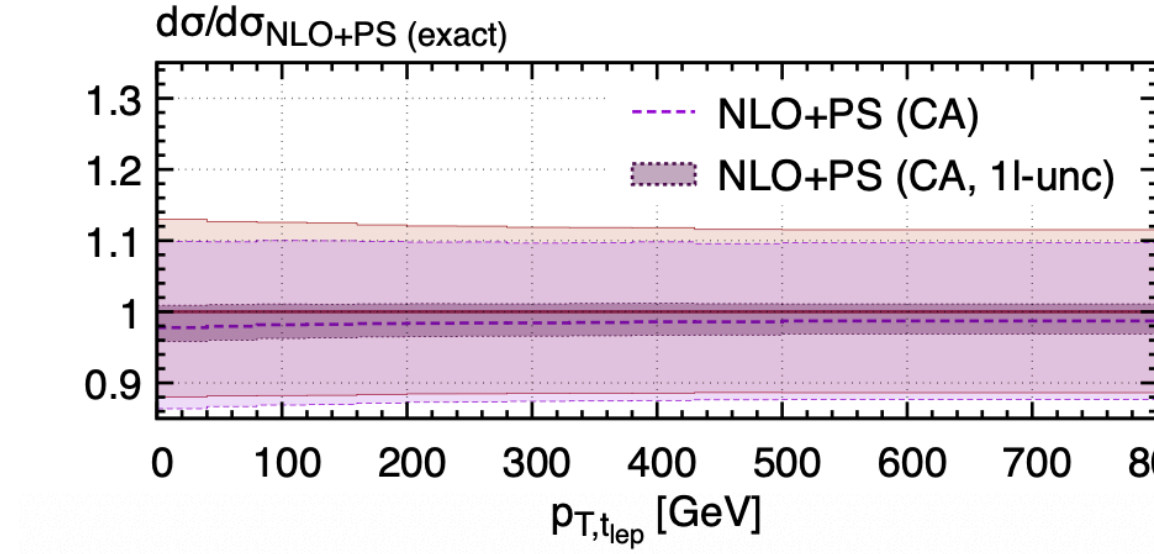
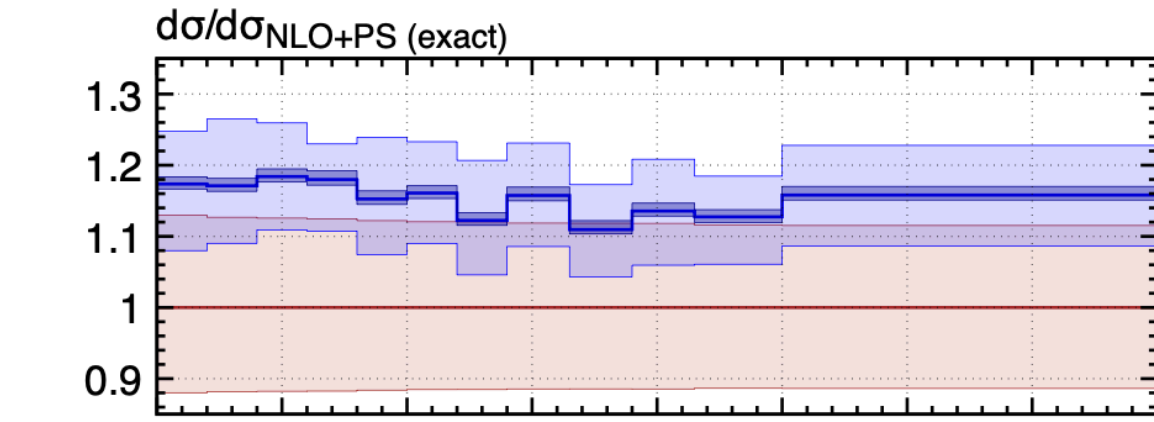
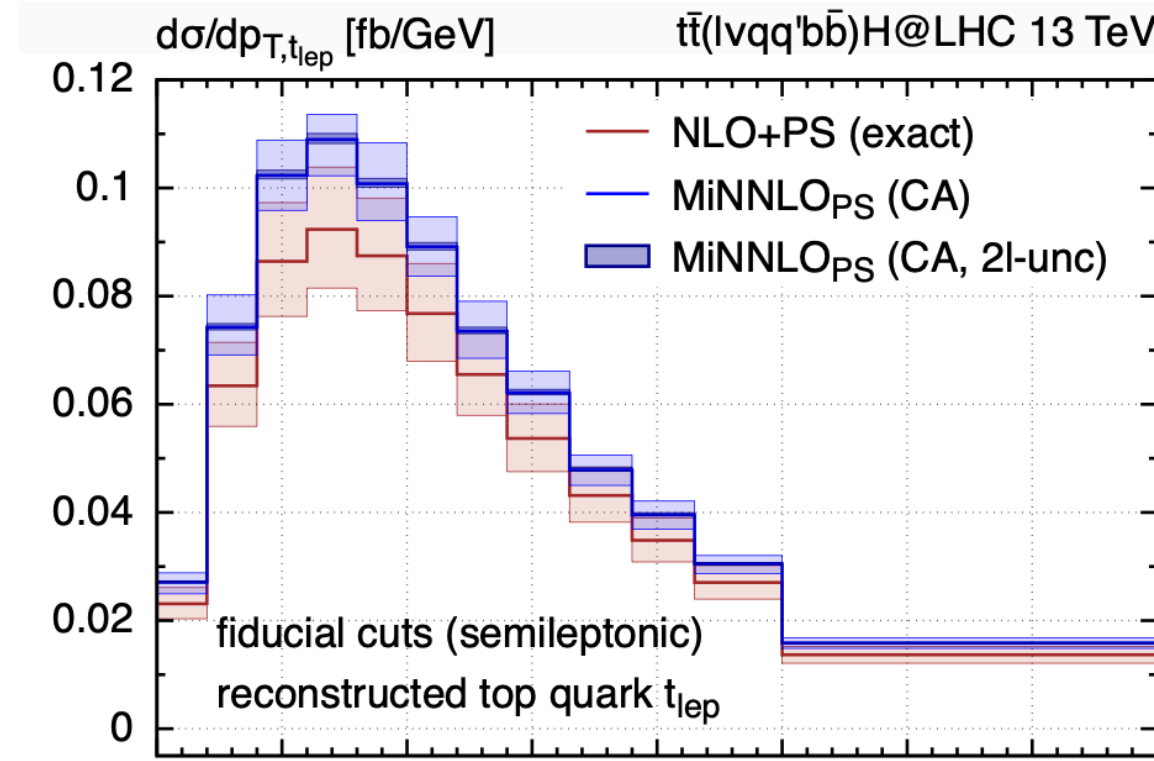
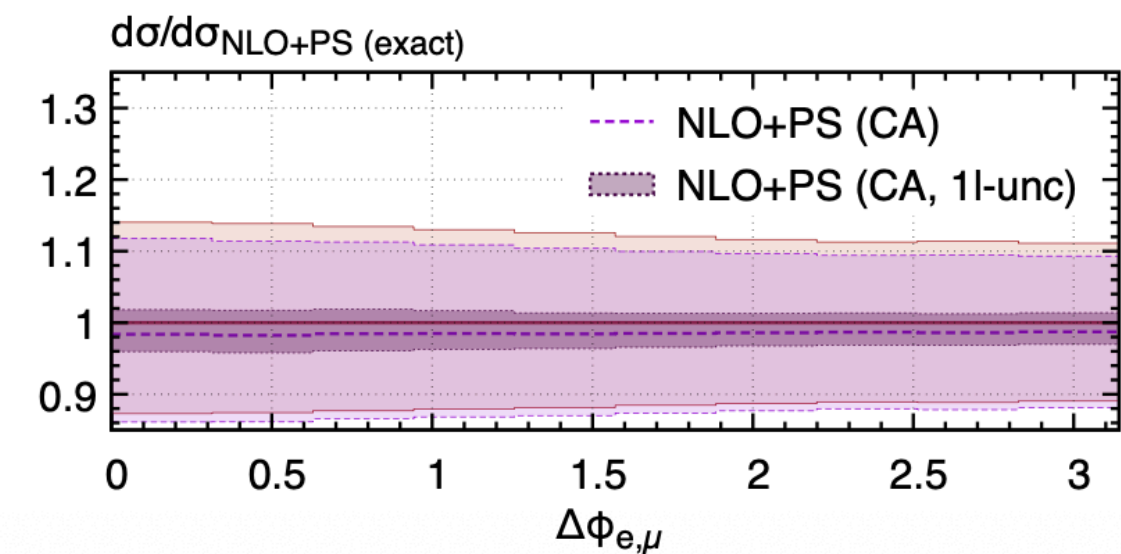
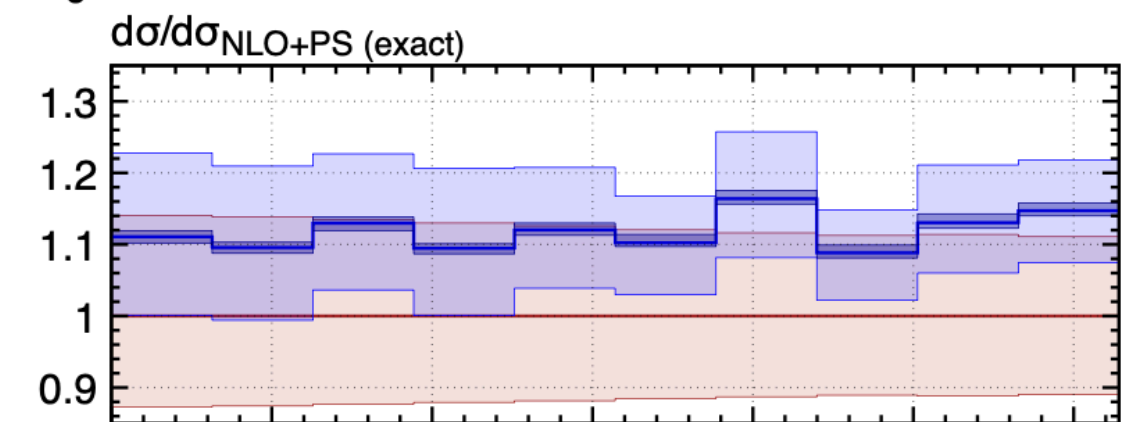
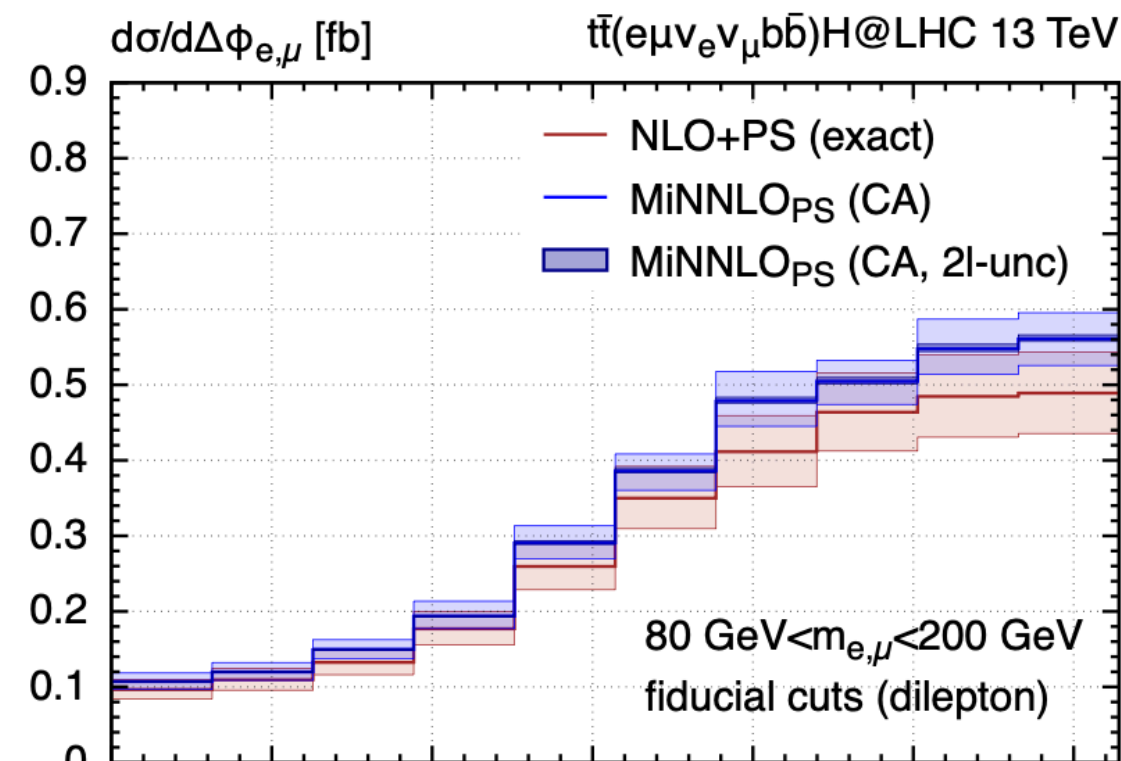
Fiducial analyses in the 2 b-jet and 2 light-jet regions

- Higgs observables in exp.-inspired fiducial regions

$$p_{T,\ell} > 30 \text{ GeV}, |\eta_\ell| < 2.4$$

$$p_{T,j} > 25 \text{ GeV}, |\eta_j| < 2.5$$

- top-quarks reconstructed with **distance-to-the-resonance** algorithm (e.g. from the $p_{T,t}$ leptonic decay)



MAIN MESSAGES

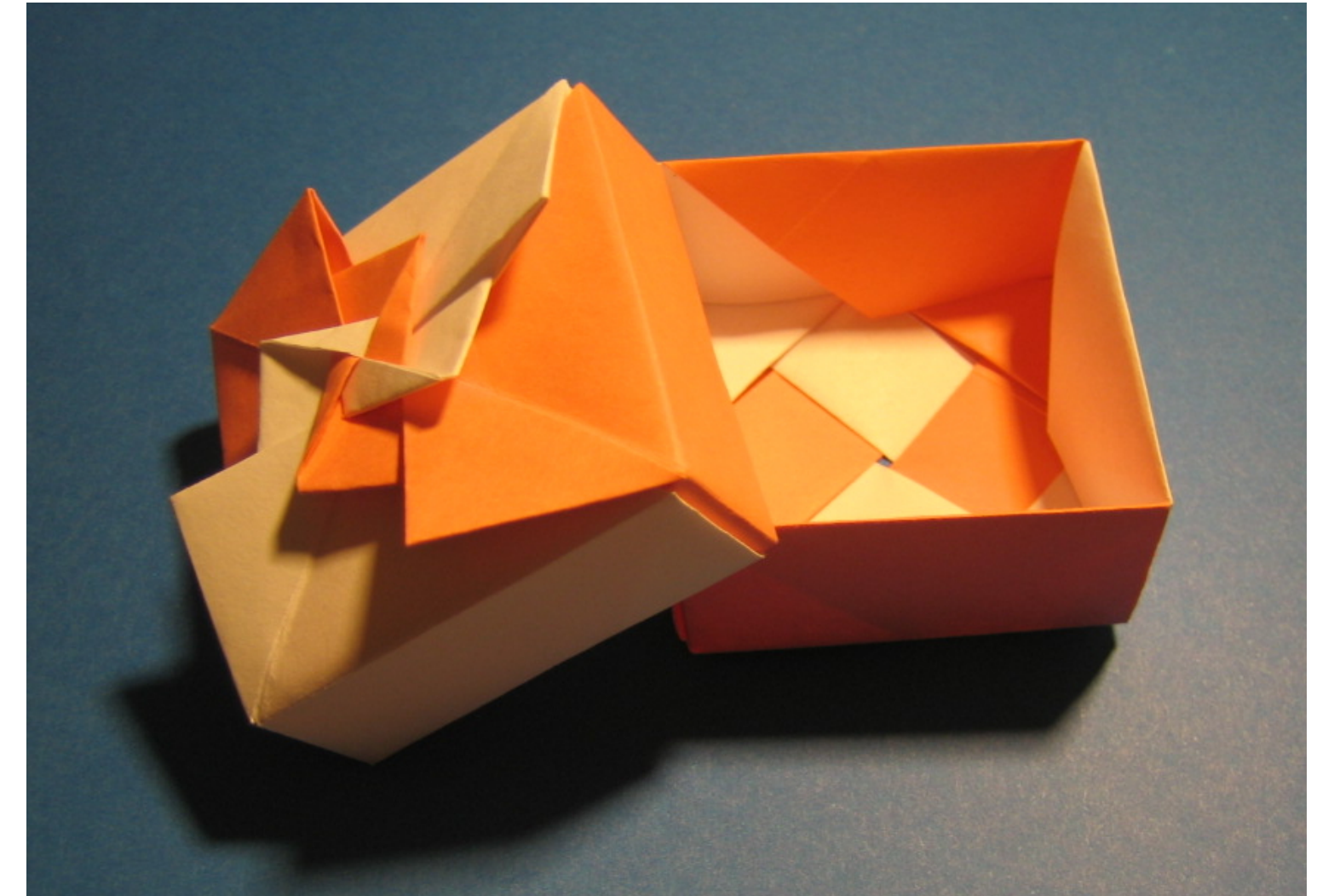
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- ▶ Flexible generators can be built on this principle, allowing for precise predictions (validation campaigns crucial to access meaningful uncertainties related to the procedure).
- ▶ Studies of differential distributions are possible with a fully general and efficient generator (including off-shell effects in different fiducial phase spaces).
- ▶ Public release of the code for further experimental and theoretical investigations

PUBLIC RELEASE

1. We release the POWHEG-BOX-RES/ttHJ code, referred to as the ttH-MiNNLO event generator (<http://gitlab.com/POWHEG-BOX/RES>)
2. The code can be downloaded by following the instructions on the POWHEG-BOX website: <http://powhegbox.mib.infn.it>. To obtain the implementation, navigate to the POWHEG-BOX-RES git directory and execute

```
$ git submodule update -- remote -- init ttHJ
```

3. All required dependencies are documented in the manual located at ttHJ/Docs/manual ttH-MiNNLO.pdf
4. Native analysis modules be used directly within POWHEG using the LHE files or together with the Pythia8 parton shower



MAIN MESSAGES

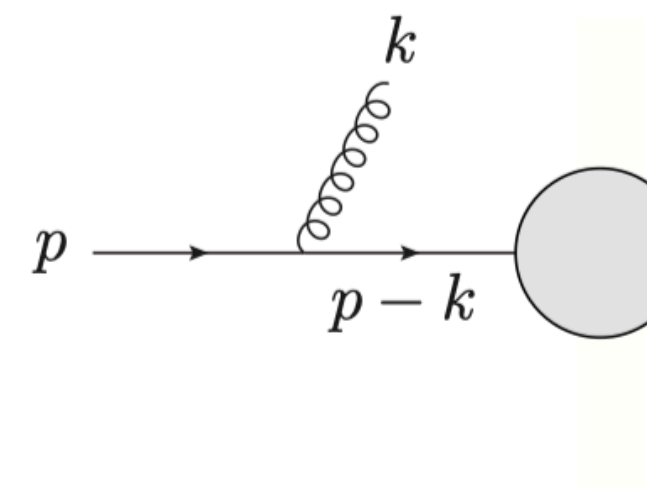
- ▶ $t\bar{t}H$ crucial for precise determination of Higgs-top Yukawa, Higgs properties, top-quark precision measurements, new physics searches.
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NESTED SOFT-COLLINEAR SUBTRACTION 2.0

IR singularities

Real corrections:

- Singularities **arising from unresolved radiation** after integration over full phase space of radiated parton
- Goal: **extract IR singularities without integrating** over the resolved phase space → obtain **fully differential prediction**



$$\sim \frac{1}{(p-k)^2} = \frac{1}{2E_p E_k (1 - \cos \theta)} \xrightarrow[E_k \rightarrow 0 \text{ or } \theta \rightarrow 0]{} \infty.$$

$$\int \frac{d^{d-1}k}{(2\pi)^{d-1} 2E_k} |M(\{p\}, k)|^2 \underset[E_k \rightarrow 0 \text{ or } \theta \rightarrow 0]{\sim} \int \frac{dE_k}{E_k^{1+2\epsilon}} \frac{d\theta}{\theta^{1+2\epsilon}} \times |M(\{p\})|^2 \sim \frac{1}{4\epsilon^2}.$$

→ **Unresolved limits are universal and known** (even at N3LO) → a **general procedure is in principle feasible**

$$\int \text{[diagram]} d\Phi_g = \underbrace{\int \left[\text{[diagram]} - \text{[diagram]} \right] d\Phi_g}_{\substack{\text{Finite in } d=4 \\ \text{integrable numerically}}} + \underbrace{\int \text{[diagram]} d\Phi_g}_{\substack{\text{exposes the same } 1/\epsilon \text{ poles as} \\ \text{the virtual correction}}}$$

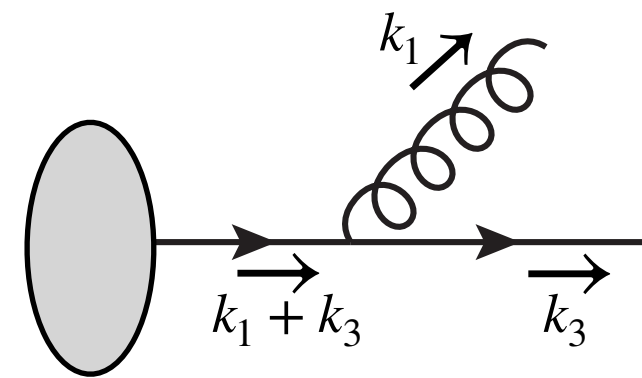
↓ Counterterm
↓ Integrated counterterm

Subtraction: conceptually non-trivial, but if local and analytic then extremely versatile and numerically stable

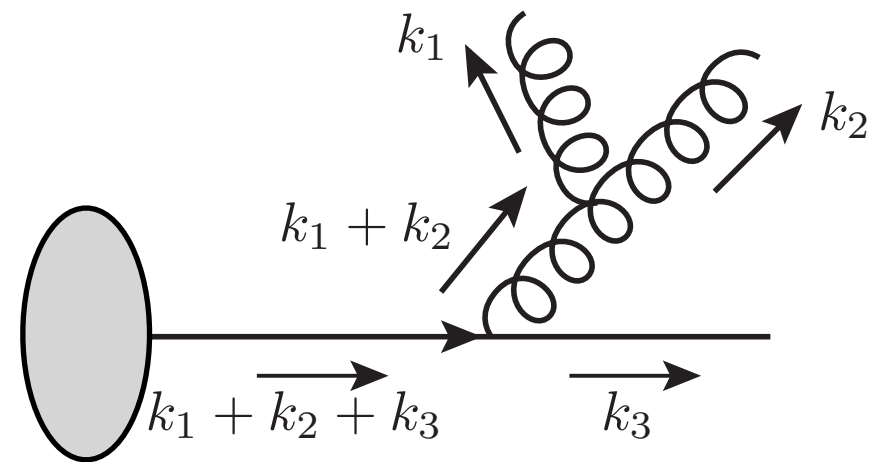
IR singularities

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- Singularities **arising from unresolved radiation** after integration over full phase space of radiated parton
- Goal: **extract IR singularities without integrating** over the resolved phase space → obtain **fully differential prediction**



$$\sim \frac{1}{(k_1 + k_3)^2} \sim \frac{1}{E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3)} \quad \begin{cases} E_1 \rightarrow 0 \\ \vec{n}_1 \parallel \vec{n}_3 \end{cases} \quad \begin{matrix} \text{soft} \\ \text{collinear} \end{matrix}$$



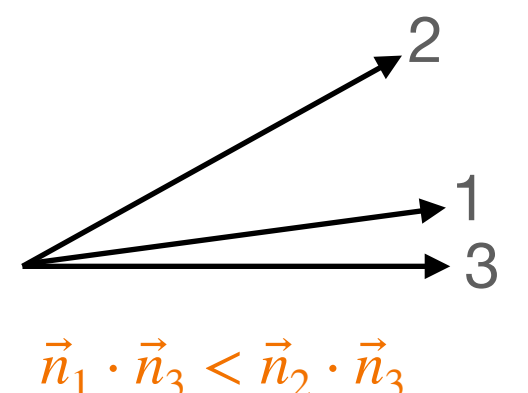
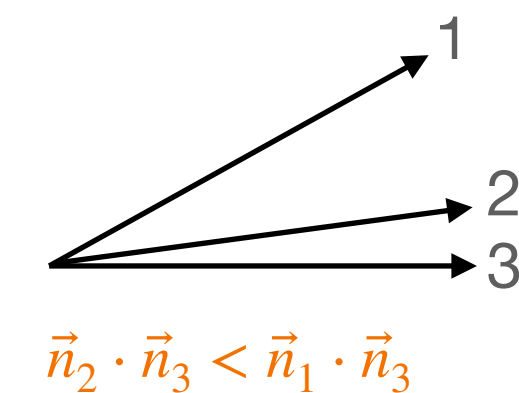
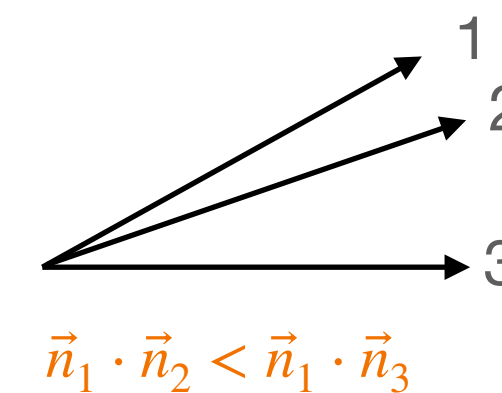
$$\sim \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2)} \frac{1}{E_1 E_2 (1 - \vec{n}_1 \cdot \vec{n}_2) + E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3) + E_2 E_3 (1 - \vec{n}_2 \cdot \vec{n}_3)}$$

$$E_1 \rightarrow 0 \quad E_2 \rightarrow 0 \quad E_1, E_2 \rightarrow 0$$

$$\vec{n}_1 \parallel \vec{n}_2 \quad \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3$$

Strongly-ordered configurations have also to be included:

$$E_1 \ll E_2, \quad E_2 \ll E_1$$



Well established schemes at NLO

- **Catani-Seymour (CS)** [\[9602277\]](#)
- **Frixione-Kunst-Signer (FKS)** [\[9512328\]](#)
- Nagy-Soper [\[1012.4948\]](#)

Currently implemented in full generality in fast and efficient NLO generators
[Gleisberg, Krauss '07, Frederix, Gehrmann, Greiner '08, Hasegawa, Moch, Uwer '09, Frederix, Frixione, Maltoni, Stelzer '09, Alioli, Nason, Oleari, Re '10, Reuter et al. '16]

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What about NNLO?

It seems that we have all the necessary ingredients:

NNLO kernels available ~ 20 years ago

[Catani, Grazzini '98, '99, '00, Kosower '99, Bern, Del Duca,
Kilgore, Schmidt '99 ...]



The recipe for a subtraction scheme seems to be known,
and involves several well-defined steps:

- clear understanding of which **singular configurations** do actually contribute
- define simplified versions of the matrix element squared to be used in the subtraction terms,
- understanding how to deal with multiple radiators and overlapping singularities (first time at NNLO),
- find a way to **integrate the subtraction terms** in d-dimensions.



Well established schemes at NLO

- Catani-Seymour (CS) [\[9602277\]](#)
- Frixione-Kunst-Signer (FKS) [\[9512328\]](#)
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What about NNLO?

**Many “final” dishes
come out, and they look
all quite different**



Many schemes are available:

Antenna [\[Gehrmann-De Ridder et al. 0505111\]](#)

ColorfullNNLO [\[Del Duca et al. 1603.08927\]](#)

Nested soft-collinear [\[Caola et al. 1702.01352\]](#)

STRIPPER [\[Czakon 1005.0274\]](#)

Analytic Analytic Sector [\[Magnea et al. 1806.09570\]](#)

Geometric IR subtraction [\[Herzog 1804.07949\]](#)

Unsubtraction [\[Sborlini et al. 1608.01584\]](#)

FDR [\[Pittau, 1208.5457\]](#)

Universal Factorisation [\[Stermann et al. 2008.12293\]](#)



Disclaimers:

- ❖ This talk is **not** about a **new subtraction method**
- ❖ On the contrary, it is a **“fresh look” at a pre-existing method** (**nested soft-collinear subtraction*** [Caola, Melnikov, Röntsch '17]) that we believe features many desirable properties that are non-trivial to enforce, especially at NNLO.
- ❖ The aim is threefold:
 - Improve our understanding on the **interplay occurring among different terms** that arise in intermediate steps
→ leading to simpler and more **compact final results**, minimising the brute-force evaluation of the counterterms, and exposing crucial cancellations.
 - Prove that the original framework is robust enough to **tackle processes with arbitrary multiplicity**.
 - Provide **final formulas** that can be adapted to **any QCD process** (i.e. treating the number of partons as a free parameter, not using any process-specific simplification) and implemented straightforwardly in **any numerical framework**.

(*) Already applied to outstanding phenomenological studies.

Nested soft-collinear subtraction at NNLO: generalities [Caola, Melnikov, Röntsche 1702.01352]

Extension of **FKS subtraction** [Frixione, Kunst, Signer 9512328] to **NNLO** and inspired by **STRIPPER** [Czakon 1005.0274]

- Exploit colour-coherence to discard interplay between soft and collinear

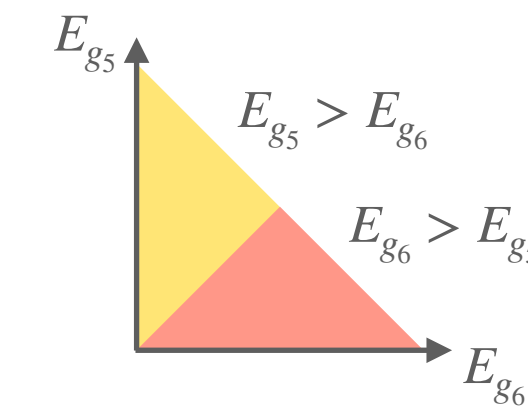
→ subtract soft limits first, then collinear

- Define subtraction terms in 3 steps:

- Globally remove double soft singularities
- Globally remove single soft singularities [using energy ordering]
- FKS partition and sectoring to treat the minimum number of collinear singularities at a time

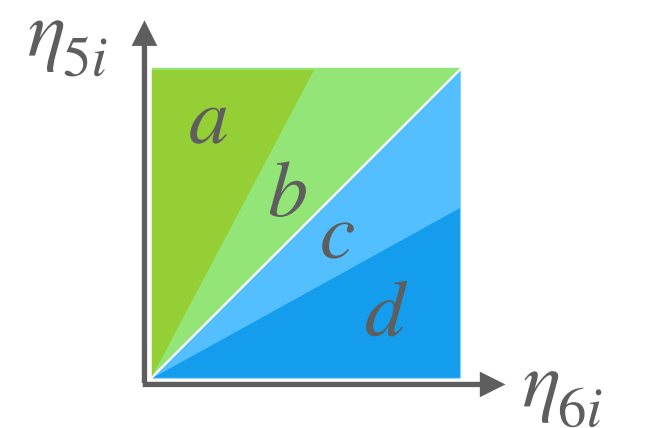
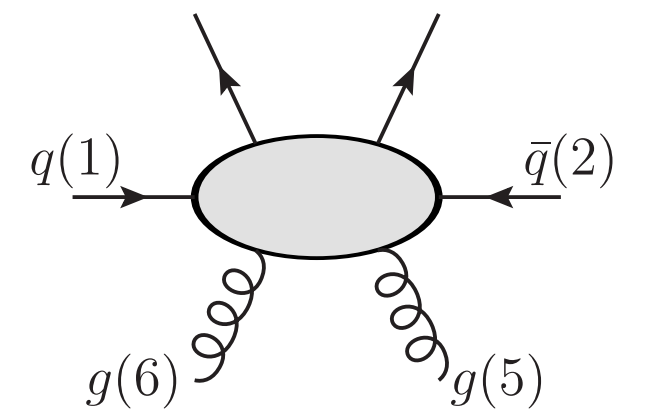
- Integrate subtraction terms analytically using Reverse Unitarity [Anastasiou, Melnikov '02]:
map phase space integrals onto loop integrals [Caola, Delto, Frellesvig, Melnikov '18, '19]

“nested approach”



$$1 = \sum_{i,j} \omega^{i5,j6}$$

$$\omega^{5i,6i} = \omega^{5i,6i} (\theta_a + \theta_b + \theta_c + \theta_d)$$



Notation and generalities:

❖ **General principle** and terminology:

$$\int \left[\text{diagram} \right] d\Phi_g = \underbrace{\int \left[\text{diagram}_1 - \text{diagram}_2 \right] d\Phi_g}_{\text{Finite in d=4}} + \underbrace{\int \text{diagram}_3 d\Phi_g}_{1/\epsilon \text{ poles}}$$

Subtraction term
Integrated subtraction term

❖ **Skeleton** of the subtraction procedure for DIS [Asteriadis, Caola, Melnikov, Röntsch '19]:

- Extract **double soft** singularities **first** (*global*)
($E_5 \sim E_6 \rightarrow 0$)

$$I = (I - \mathcal{S}) + \mathcal{S}$$

$$\left| \text{diagram}_1 \right|^2 = (I - \mathcal{S}) \times \left| \text{diagram}_2 \right|^2 + \mathcal{S} \left| \text{diagram}_3 \right|^2$$

- Then single soft (**energy ordering** of the emissions to reduce the number of singularities)

$$I = (I - S_6) + S_6$$

$$(I - \mathcal{S}) \times \left| \text{diagram}_1 \right|^2 = (I - \mathcal{S})(I - S_6) \times \left| \text{diagram}_2 \right|^2 + (\text{Subtraction terms})$$

- Collinear singularities (*local*): **partition function** + **sectoring** [separate overlapping singularities]

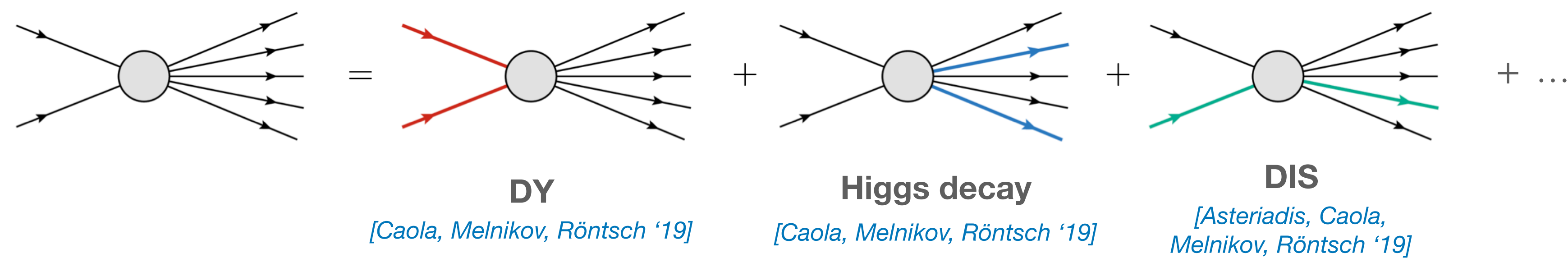
$$(I - \mathcal{S})(I - S_6) \times \left| \text{diagram}_1 \right|^2 = \sum_{ij} (I - \mathcal{S})(I - S_6)(I - \mathcal{C})(I - C_{61}) \times w_i \times \theta_j \times \left| \text{diagram}_2 \right|^2 + (\text{Subtraction terms})$$

triple collinear singularity *double collinear singularity; e.g. (6//1)* *sector; angular ordering* *partition function*

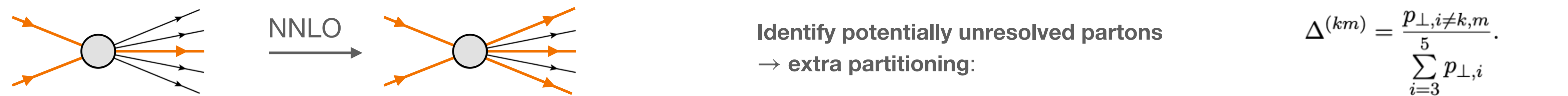
[figures courtesy of K. Asteriadis]

What we have, what we want

❖ **Bottom-up approach:** building $pp \rightarrow N_{\text{jets}}$ starting from simpler processes. **Application to low-multiplicity** processes worked out straightforwardly

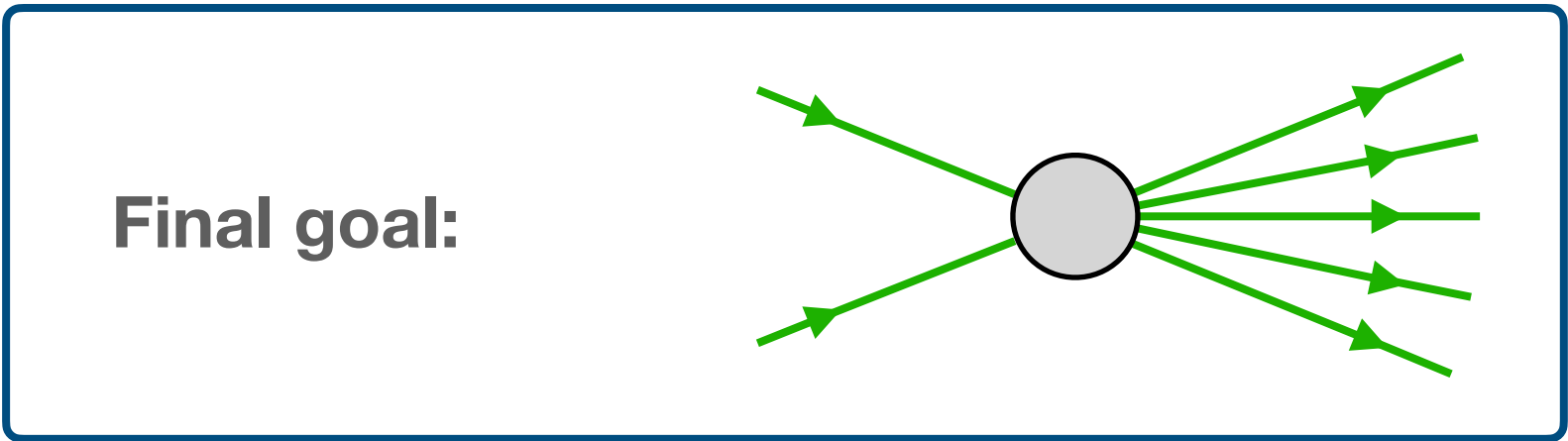


❖ Natural generalisation: **V+j @ NNLO QCD**



This is the highest Born-level multiplicity that allows for a closed colour algebra and for expressing the color of LO-configurations entirely via Casimir operators

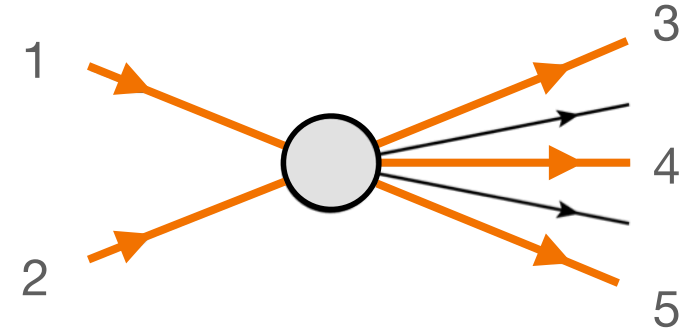
$$0 = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 \implies \begin{aligned} 2\mathbf{T}_1 \cdot \mathbf{T}_2 &= C_3 - C_1 - C_2, \\ 2\mathbf{T}_2 \cdot \mathbf{T}_3 &= C_1 - C_2 - C_3, \\ 2\mathbf{T}_3 \cdot \mathbf{T}_1 &= C_2 - C_3 - C_1. \end{aligned}$$



Application to V+j

❖ In the case of gluon final state the formal expression for the **regularised double-real** correction is quite simple.

❖ This can be done because we know how to deal with **multiple radiators** [partitioning, energy ordering]



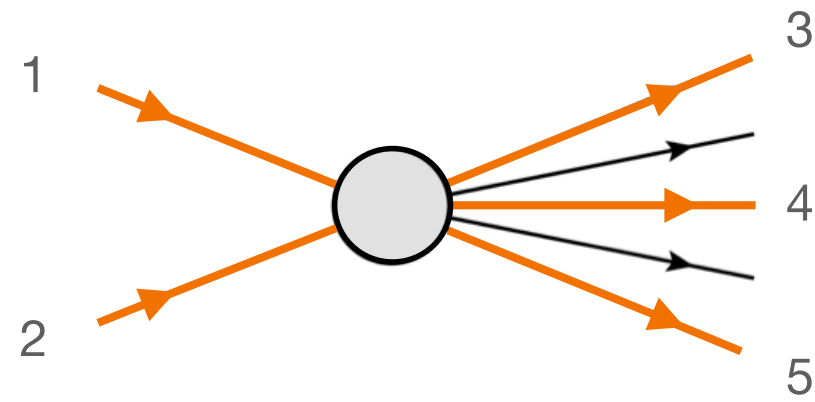
$$\begin{aligned} \frac{1}{3!} \langle F_{\text{LM}}(1_q, 2_{\bar{q}}; 3_g, 4_g, 5_g) \rangle = & \langle S_{45} \Delta^{(45)} F_{\text{LM}}^{4>5} \rangle + \langle (I - S_4) S_5 \Delta^{(45)} F_{\text{LM}}^{4>5} \rangle \\ & + \left\langle (I - S_{45})(I - S_5) \left\{ \sum_{i \in \text{TC}} \left[\Theta^{(a)} C_{45,i} (I - C_{5i}) + \Theta^{(b)} C_{45,i} (I - C_{45}) \right. \right. \right. \\ & \quad \left. \left. + \Theta^{(c)} C_{45,i} (I - C_{4i}) + \Theta^{(d)} C_{45,i} (I - C_{45}) \right] \omega_{4i5i} \right\} \Delta^{(45)} F_{\text{LM}}^{4>5} \right\rangle \\ & - \left\langle (I - S_{45})(I - S_5) \sum_{(ij) \in \text{DC}} C_{4i} C_{5j} \omega_{4i5j} \Delta^{(45)} F_{\text{LM}}^{4>5} \right\rangle \\ & + \left\langle (I - S_{45})(I - S_5) \left\{ \sum_{i \in \text{TC}} \left[\Theta^{(a)} C_{5i} + \Theta^{(b)} C_{45} + \Theta^{(c)} C_{4i} + \Theta^{(d)} C_{45} \right] \omega_{4i5i} \right. \right. \\ & \quad \left. \left. + \sum_{(ij) \in \text{DC}} [C_{4i} + C_{5j}] \omega_{4i5j} \right\} \Delta^{(45)} F_{\text{LM}}^{4>5} \right\rangle \quad \text{Subtraction terms} \\ & + \left\langle (I - S_{45})(I - S_5) \left\{ \sum_{i \in \text{TC}} \left[\Theta^{(a)} (I - C_{45,i}) (I - C_{5i}) + \Theta^{(b)} (I - C_{45,i}) (I - C_{45}) \right. \right. \right. \\ & \quad \left. \left. + \Theta^{(c)} (I - C_{45,i}) (I - C_{4i}) + \Theta^{(d)} (I - C_{45,i}) (I - C_{45}) \right] \omega_{4i5i} \right. \right. \\ & \quad \left. \left. + \sum_{(ij) \in \text{DC}} (I - C_{4i}) (I - C_{5j}) \omega_{4i5j} \right\} \Delta^{(45)} F_{\text{LM}}^{4>5} \right\rangle \quad \text{Fully regulated term} \end{aligned}$$

- The **bookkeeping** becomes immediately **cumbersome** → large number of subtraction terms.
- Calculating all **subtraction terms separately** may **hide a number of simplifications** that can occur **before explicit evaluation**.
- Writing color-correlations as Casimir operators in the **intermediate steps facilitates the calculation, BUT** leads to **non-trivial generalisation** to n-partons.

$$\begin{aligned} (ij) \in \text{DC} &\longrightarrow (ij) \in \{(12), (13), (21), (23), (31), (32)\} \\ i \in \text{TC} &\longrightarrow i \in \{1, 2, 3\}. \end{aligned}$$

Application to V+j: the lesson

- ❖ The expression of **double-real** poles and finite remainder results in **large, non-transparent bunch of terms** (what if we now want to change the flavour of final-state partons? Do we have to start from the beginning?)

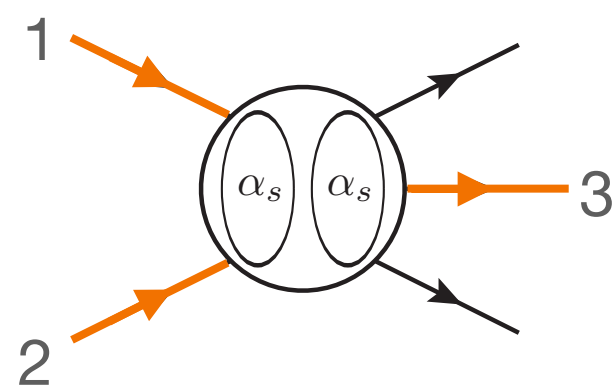


```
Series[RRpoles, {ε, 0, -1}]

1/4 ε^4 (3 asontwopi^2 CA^2 FLM[p1q, p2q, p3g] + 10 asontwopi^2 CA CF FLM[p1q, p2q, p3g] + 2 asontwopi^2 CF^2 FLM[p1q, p2q, p3g] +
5 asontwopi^2 CF^2 delta[1-z] x FLM[p1q, z p2q, p3g, z] + 5 asontwopi^2 CF^2 delta[1-z] x FLM[z p1q, p2q, p3g, z] -
4 asontwopi^2 CF^2 delta[1-z1] x delta[1-z2] x FLM[z1 p1q, z2 p2q, p3g, z1, z2]) +
77 asontwopi^2 CA^2 FLM[p1q, p2q, p3g] + 110 asontwopi^2 CA CF FLM[p1q, p2q, p3g] + 72 asontwopi^2 CA^2 L3 FLM[p1q, p2q, p3g] + 36 asontwopi^2 CA CF FLM[p1q, z p2q, p3g, z] + ... 100 ...
+
1381/72 asontwopi^2 CA^2 FLM[p1q, p2q, p3g] + 67/4 asontwopi^2 CA CF FLM[p1q, p2q, p3g] + ... 509 ... + 1/2 asontwopi^2 CA CF delta[1-z] x FLM[z p1q, p2q, p3g, z] PolyLog[2, 1-eta[2,3]] +
+
3151/27 asontwopi^2 CA^2 FLM[p1q, p2q, p3g] + 2813/36 asontwopi^2 CA CF FLM[p1q, p2q, p3g] + 265/18 asontwopi^2 CA^2 L3 FLM[p1q, p2q, p3g] - 11/2 ... 3 ... FLM[p1q, p2q, p3g] + ... 2016 ... + 0[ε]^0

large output show less show more show all set size limit...
```

- ❖ Interestingly, the **IR structure of the double-virtual** corrections does not reflect the same complexity [\[Catani '98\]](#)



$$\begin{aligned} \langle \mathcal{M} | \mathcal{M} \rangle_{\alpha_s^2} = & \left\langle \mathcal{M}_0 \left| \frac{1}{2} I_1^2(\epsilon) + \frac{1}{2} (I_1^\dagger(\epsilon))^2 + I_1^\dagger(\epsilon) I_1(\epsilon) + (\mathcal{H}_2 + \mathcal{H}_2^\dagger) \right| \mathcal{M}_0 \right\rangle \\ & + \left\langle \mathcal{M}_0 \left| -\frac{\beta_0}{\epsilon} (I_1(\epsilon) + I_1^\dagger(\epsilon)) + c_\epsilon \left(\frac{\beta_0}{\epsilon} + K \right) (I_1(2\epsilon) + I_1^\dagger(2\epsilon)) \right| \mathcal{M}_0 \right\rangle \\ & + 2\text{Re}[\langle \mathcal{M}_0 | I_1(\epsilon) + I_1^\dagger(\epsilon) | \mathcal{M}_1^{\text{fin}} \rangle] + 2\text{Re}[\langle \mathcal{M}_0 | \mathcal{M}_2^{\text{fin}} \rangle] + \langle \mathcal{M}_1^{\text{fin}} | \mathcal{M}_1^{\text{fin}} \rangle. \end{aligned}$$

```
Series[VVpoles, {ε, 0, -1}]

asontwopi^2 (CA + 2 CF)^2 FLM[p1q, p2q, p3g] +
+
asontwopi^2 (CA + 2 CF) FLM[p1q, p2q, p3g] (12 CF + 7 β0 + 4 (CA - 2 CF) Log[s12/mu2] - 4 CA (Log[s13/mu2] + Log[s23/mu2])) + 1/ε^2 asontwopi^2
+
(-CA - 2 CF) FLVfin[p1q, p2q, p3g] + FLM[p1q, p2q, p3g] (1/72 (-2 CA CF (67 + 9 π^2) + CA^2 (-67 + 33 π^2) + 20 CA Nf TR + 4 CF (CF (81 - 42 π^2) + 10 Nf TR)) +
9 CF β0/2 + β0^2 + 1/4 (8 CF (-CA + 2 CF) Log[s12/mu2]^2 + 4 CA (CA + CF) Log[s13/mu2]^2 + Log[s13/mu2] (-3 CA (CA + 6 CF) - 4 (2 CA + CF) β0 + 4 CA^2 Log[s23/mu2]) + Log[s23/mu2]
(-3 CA (CA + 6 CF) - 4 (2 CA + CF) β0 + 4 CA (CA + CF) Log[s23/mu2]) - 2 (CA - 2 CF) Log[s12/mu2] (-3 (CA + 4 CF + β0) + 2 CA (Log[s13/mu2] + Log[s23/mu2])))) +
1/ε asontwopi^2 (FLVfin[p1q, p2q, p3g] (-3 CF - β0 - (CA - 2 CF) Log[s12/mu2] + CA (Log[s13/mu2] + Log[s23/mu2])) +
1/864 FLM[p1q, p2q, p3g] (3 CA^2 (60 + 227 π^2) + 2 CA (CF (-1922 + 333 π^2) - 2 Nf (232 + 3 π^2) TR + 3 (-268 + 51 π^2) β0) +
4 (-27 CF^2 (3 + 52 π^2) + 2 CF Nf (184 + 9 π^2) TR - 315 CF π^2 β0 + 40 Nf TR (2 Nf TR + 3 β0)) -
12 (24 (CA - 4 CF) (CA - 2 CF) Log[s12/mu2]^3 + 24 CA (2 CA + CF) Log[s13/mu2]^3 + 9 Log[s13/mu2]^2 (-9 CA (CA + 2 CF) - 2 (5 CA + 2 CF) β0 + 4 CA^2 Log[s23/mu2]) -
18 (CA - 2 CF) Log[s12/mu2]^2 (3 CA - 2 (12 CF + β0) + 2 CA (Log[s13/mu2] + Log[s23/mu2])) + 2 (CA - 2 CF) Log[s12/mu2]
(67 CA + 3 (CA + 28 CF) π^2 - 2 (81 CF + 10 Nf TR + 27 β0) + 9 (9 CA + 2 β0) Log[s13/mu2] - 18 CA Log[s13/mu2]^2 + 9 Log[s23/mu2] (9 CA + 2 β0 - 2 CA Log[s23/mu2])) +
2 Log[s13/mu2] (CA (-67 CA + 81 CF + 3 (5 CA - 16 CF) π^2 + 20 Nf TR) + 27 (CA + 2 CF) β0 + 18 β0^2 + 18 CA Log[s23/mu2] (-3 CA - 2 β0 + CA Log[s23/mu2])) +
Log[s23/mu2] (2 CA (-67 CA + 81 CF + 3 (5 CA - 16 CF) π^2 + 20 Nf TR) + 54 (CA + 2 CF) β0 + 36 β0^2 +
3 Log[s23/mu2] (-27 CA (CA + 2 CF) - 6 (5 CA + 2 CF) β0 + 8 CA (2 CA + CF) Log[s23/mu2])) + 6 (CA^2 - 62 CA CF + 88 CF^2) Zeta[3])) + 0[ε]^0
```


Can we identify structures **early on** in the calculations so that cancellation of divergences can be seen “by eye”, even for a **generic process**?

Main idea: look at the pole structure of the virtual corrections to infer similar structures for the subtraction terms

→ reverses the standard logic guiding the construction of most established infrared subtraction schemes*

Case study: $q\bar{q} \rightarrow X + Ng$



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A fresh look at the nested soft-collinear subtraction scheme: NNLO QCD corrections to N -gluon final states in $q\bar{q}$ annihilation

Federica Devoto^a, Kirill Melnikov^b, Raoul Röntsch^c, Chiara Signorile-Signorile^{b,d,e} and Davide Maria Tagliabue^c

First generalisation: $gq \rightarrow X + (N - 1)g + q$



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Towards a general subtraction formula for NNLO QCD corrections to processes at hadron colliders: final states with quarks and gluons

Federica Devoto^a, Kirill Melnikov^b, Raoul Röntsch^c, Chiara Signorile-Signorile^d, Davide Maria Tagliabue^{b,c} and Matteo Tresoldi^b

Fully general: $pp \rightarrow X + Nj$

Integrated subtraction terms and finite remainders for arbitrary processes with massless partons at colliders in the nested soft-collinear subtraction scheme

[Submitted on 10 Sep 2025]

Federica Devoto^a, Kirill Melnikov^b, Raoul Röntsch^c, Chiara Signorile-Signorile^d, Davide Maria Tagliabue^b, Matteo Tresoldi^b

(*) similar idea also explored in the context of *antenna subtraction* [Gehrmann, Glover, Marcoli '23] and *local analytic sector subtraction* [Magnea, Milloy, CSS, Torrielli '24]

Warm up @NLO: $q\bar{q} \rightarrow X + Ng$

$$2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = d\hat{\sigma}_{ab}^{\text{V}} + d\hat{\sigma}_{ab}^{\text{R}} + d\hat{\sigma}_{ab}^{\text{pdf}}$$

- 1) Virtual corrections:
color-correlations, elastic terms

$$I_V(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2} \sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

- 2) Real corrections: soft: color-correlations, elastic terms

$$I_S(\epsilon) = -\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j)$$

$$I_V(\epsilon) + I_S(\epsilon)$$

- Highest pole trivially cancels
- Color correlations cancel
- Remnant elastic single pole

Warm up @NLO: $q\bar{q} \rightarrow X + Ng$

$$2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = d\hat{\sigma}_{ab}^{\text{V}} + d\hat{\sigma}_{ab}^{\text{R}} + d\hat{\sigma}_{ab}^{\text{pdf}}$$

- 1) Virtual corrections:
color-correlations, elastic terms

$$I_V(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2} \sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

- 2) Real corrections:

soft: color-correlations, elastic terms

$$I_S(\epsilon) = -\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j)$$

$$I_V(\epsilon) + I_S(\epsilon)$$

- Highest pole trivially cancels
- Color correlations cancel
- Remnant elastic single pole

hard-collinear: no color-correlations, boosts + elastic terms

$\mathcal{P}_{aa}^{\text{gen}} \otimes F_{\text{LM}}$

$$I_C(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}$$

“generalised anomalous dimensions”

$$\Gamma_{i,f_i} = \gamma_i + 2\mathbf{T}_i^2 L_i + \mathcal{O}(\epsilon)$$



$$I_T(\epsilon) = I_V(\epsilon) + I_S(\epsilon) + I_C(\epsilon) \quad \text{FINITE}$$

Warm up @NLO: $q\bar{q} \rightarrow X + Ng$

$$2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = d\hat{\sigma}_{ab}^{\text{V}} + d\hat{\sigma}_{ab}^{\text{R}} + d\hat{\sigma}_{ab}^{\text{pdf}}$$

- 1) Virtual corrections:
color-correlations, elastic terms

$$I_V(\epsilon) = \bar{I}_1(\epsilon) + \bar{I}_1^\dagger(\epsilon)$$

$$\bar{I}_1(\epsilon) = \frac{1}{2} \sum_{(ij)}^{N_p} \frac{\mathcal{V}_i^{\text{sing}}(\epsilon)}{\mathbf{T}_i^2} (\mathbf{T}_i \cdot \mathbf{T}_j) \left(\frac{\mu^2}{2p_i \cdot p_j} \right)^\epsilon e^{i\pi\lambda_{ij}\epsilon}$$

$$\mathcal{V}_i^{\text{sing}}(\epsilon) = \frac{\mathbf{T}_i^2}{\epsilon^2} + \frac{\gamma_i}{\epsilon}$$

- 2) Real corrections: soft: color-correlations, elastic terms

$$I_S(\epsilon) = -\frac{(2E_{\text{max}}/\mu)^{-2\epsilon}}{\epsilon^2} \sum_{(ij)}^{N_p} \eta_{ij}^{-\epsilon} K_{ij} (\mathbf{T}_i \cdot \mathbf{T}_j)$$

$$I_V(\epsilon) + I_S(\epsilon)$$

- Highest pole trivially cancels
- Color correlations cancel
- Remnant elastic single pole

hard-collinear: no color-correlations, boosts + elastic terms

$$\mathcal{P}_{aa}^{\text{gen}} \otimes F_{\text{LM}} \quad I_C(\epsilon) = \sum_{i=1}^{N_p} \frac{\Gamma_{i,f_i}}{\epsilon}$$

“generalised anomalous dimensions”

$$\Gamma_{i,f_i} = \gamma_i + 2\mathbf{T}_i^2 L_i + \mathcal{O}(\epsilon)$$



$$I_T(\epsilon) = I_V(\epsilon) + I_S(\epsilon) + I_C(\epsilon) \quad \text{FINITE}$$

- 3) PDFs renormalisation: no color-correlations, boosts $\hat{P}_{aa}^{(0)} \otimes F_{\text{LM}}$

4) Sum:

$$2s \, d\hat{\sigma}_{ab}^{\text{NLO}} = \frac{\alpha_s(\mu)}{2\pi} \langle I_T^{(0)} \cdot F_{\text{LM}} \rangle + \frac{\alpha_s(\mu)}{2\pi} \left[\langle \mathcal{P}_{aa}^{\text{NLO}} \otimes F_{\text{LM}} \rangle + \langle F_{\text{LM}} \otimes \mathcal{P}_{bb}^{\text{NLO}} \rangle \right] + \langle F_{\text{LV}}^{\text{fin}} \rangle + \langle \mathcal{O}_{\text{NLO}} \Delta^{(\mathbf{m})} F_{\text{LM}}(\mathbf{m}) \rangle$$

Collinear limits: symmetric case

- ❖ Consider N gluons created in a quark-antiquark annihilation: $q\bar{q} \rightarrow \{g\}_N + V$ (most singular configuration, also most symmetric)

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue '23]

- ❖ To restrict the number of unresolved configurations use **damping factors**:

$$2\hat{s} d\sigma^R = \left\langle \frac{1}{(N+1)!} F_{\text{LM}}[\{g\}_{N+1}] \right\rangle = \langle \Delta^{(\mathbf{m})} F_{\text{LM}}[\{g\}_N | \mathbf{m}] \rangle. \quad (\text{symmetry properties of } F_{\text{LM}} \rightarrow \text{only one unresolved parton})$$

- ❖ Damping factors are sensitive to IR limits

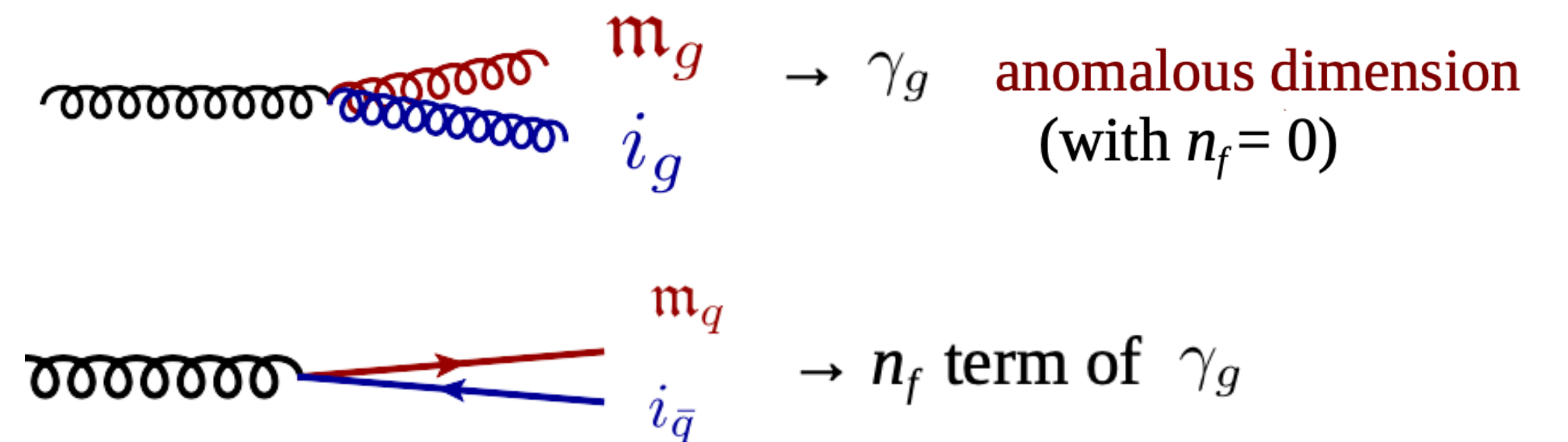
$$S_{\mathbf{m}} \Delta^{(i)} = 1 \quad C_{a\mathbf{m}} \Delta^{(i)} = 1 \quad C_{i\mathbf{m}} \Delta^{(i)} = z_{i,\mathbf{m}} = E_i / (E_i + E_{\mathbf{m}})$$

- ❖ Effects on the subtraction: **final-state collinear limits**

$$\Gamma_{i, f_{[i\mathbf{m}]} \rightarrow f_i f_{\mathbf{m}}} \sim - \int_0^1 dz \bar{S}_z z (z(1-z))^{-2\epsilon} \mathcal{P}_{f_{[i\mathbf{m}]} f_i}(z) + 2\delta_{f_{[i\mathbf{m}]} f_i} \mathbf{T}_{f_{[i\mathbf{m}]}}^2$$

Limit of damping factor

$$\Gamma_{i,g} = \Gamma_{i,g \rightarrow gg} + 2n_f \Gamma_{i,g \rightarrow q\bar{q}} = \gamma_g + 2T_g^2 L_i + \mathcal{O}(\epsilon)$$



Collinear limits: what happens when considering “asymmetries”?

- ❖ Consider less symmetric process: $gq \rightarrow q + \{g\}_{N_g} + V$

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]

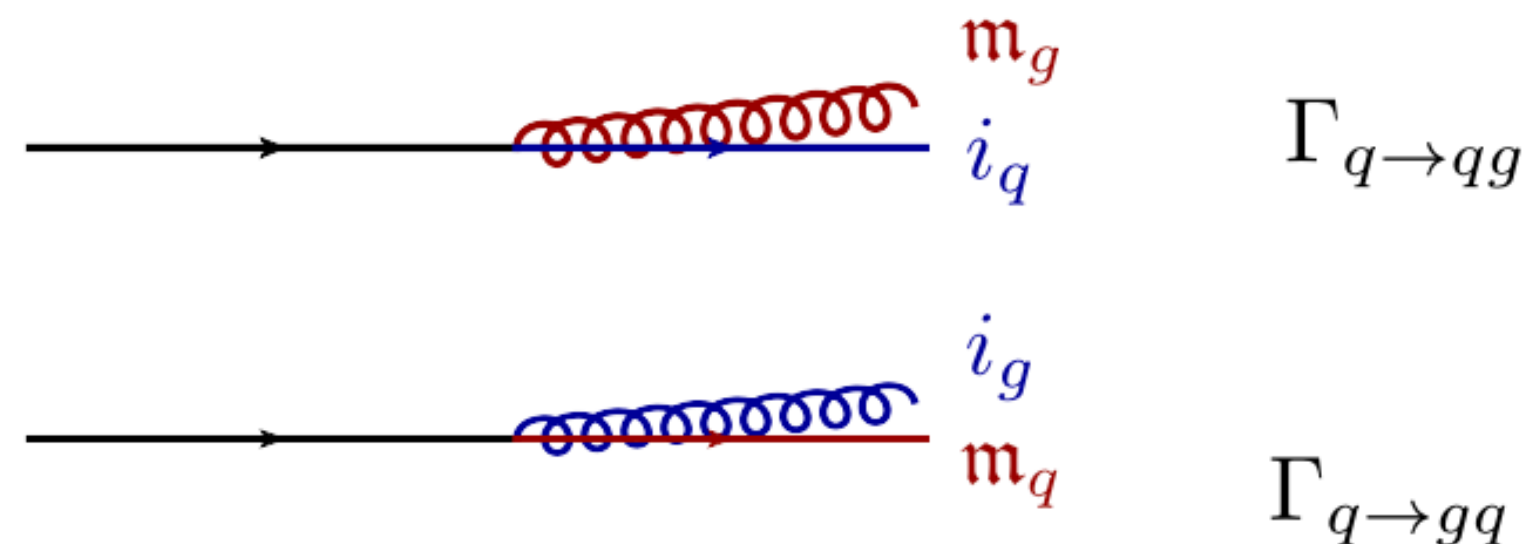
- ❖ The **damping factors** introduce two main configurations

$$2s_{ab} d\hat{\sigma}^R = \sum_i \left\langle \frac{\Delta^{(i)}}{(N_g + 1)!} F_{\text{LM}}[\{g\}_{N_g+1}, q] \right\rangle = \left\langle \Delta^{(\mathfrak{m})} F_{\text{LM}}[\{g\}_{N_g}, q | \mathfrak{m}_g] \right\rangle + \left\langle \Delta^{(\mathfrak{m})} F_{\text{LM}}[\{g\}_{N_g+1} | \mathfrak{m}_q] \right\rangle$$

Unresolved **gluon**: similar procedure as before+collinear limits wrt to quark

Unresolved **quark**: only collinear limits have to be subtracted and no soft overlap

- ❖ Final-state collinear limits: need to distinguish potentially-unresolved parton from emitter



$$\Gamma_{q \rightarrow qg} + \Gamma_{q \rightarrow gq} = \Gamma_q = -(\gamma_q + 2\mathbf{T}_i^2 L_i) + \mathcal{O}(\epsilon)$$

Due to **limit of damping factor** in integral:

$$\Gamma_{i, f_{[i\mathfrak{m}]} \rightarrow f_i f_{\mathfrak{m}}} \sim - \int_0^1 dz \bar{S}_z \textcircled{z} (z(1-z))^{-2\epsilon} \mathcal{P}_{f_{[i\mathfrak{m}]} f_i}(z) + 2\delta_{f_{[i\mathfrak{m}]} f_i} \mathbf{T}_{f_{[i\mathfrak{m}]} }^2$$

Fully general framework

- ❖ Fully generic (massless) process: $pp \rightarrow N \text{ jets} + X$

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]

- ❖ Conceptually fully understood, thanks to the experience gained in the previous cases: damping factors to divide different singular configurations, combination of different limits into universal operator, cancellations prior to evaluation...
- ❖ New element: **interplay of different partonic channels and relevant IR limits**

n enumerates all QCD final states which may contribute to the partonic process, including all combinations of flavour consistent with the initial state (a, b)

$$2s_{ab} d\hat{\sigma}_{ab}^R = \sum_n \langle F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}] \rangle$$

final state with $N + 1$ QCD partons that can be produced in the collision of parton a and b

- ❖ Symmetry properties of matrix elements+damping factors $\rightarrow$ minimal set of potentially singular configurations

$$2s_{ab} d\hat{\sigma}_{ab}^R = \sum_n \langle F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}] \rangle = \sum_n \left\langle \Delta^{(\mathfrak{m})} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathfrak{m}_g)] \right\rangle + \sum_n \sum_{\rho=1}^{n_f} \left\langle \Delta^{(\mathfrak{m})} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathfrak{m}_{q_\rho})] \right\rangle + q_\rho \rightarrow \bar{q}_\rho$$

Identify gluon/quark/antiquark as potentially unresolved and sum over all possible configurations

Fully general framework

- ❖ Fully generic (massless) process: $pp \rightarrow N \text{ jets} + X$

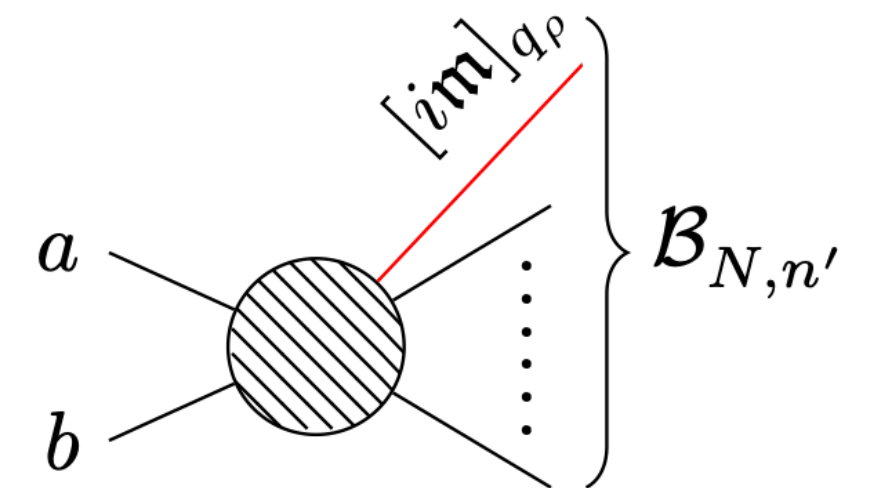
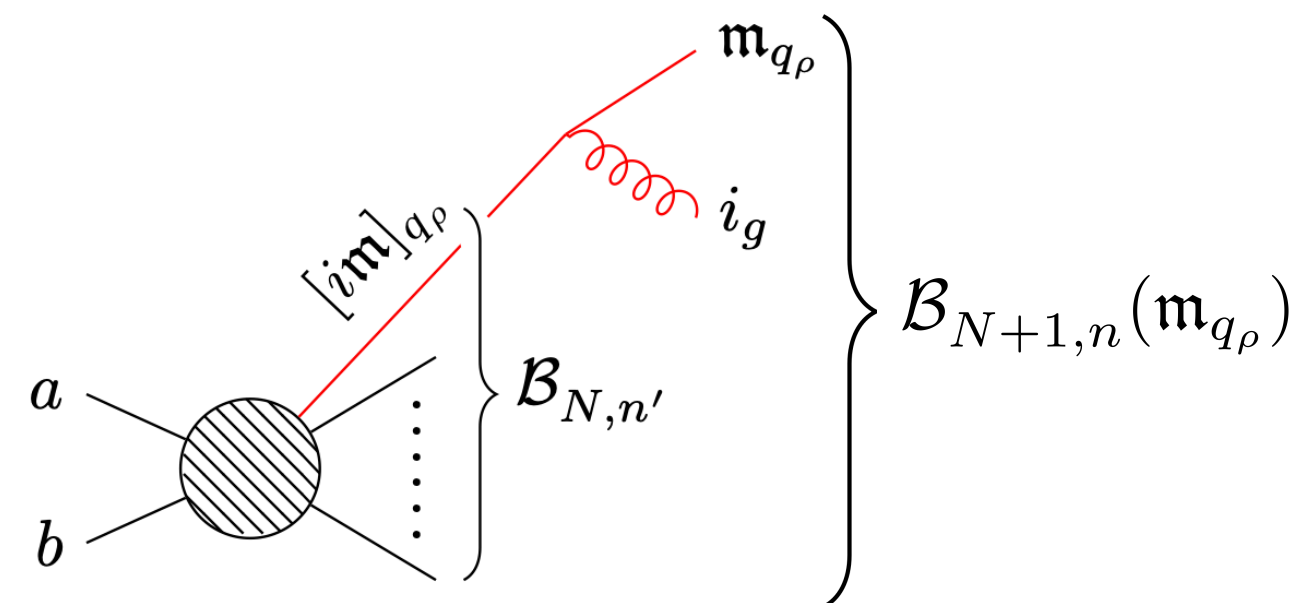
[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]

- ❖ Conceptually fully understood, thanks to the experience gained in the previous cases: damping factors to divide different singular configurations, combination of different limits into universal operator, cancellations prior to evaluation...

- ❖ New element: **interplay of different partonic channels and relevant IR limits**

$$\left\langle \bar{S}_{\mathbf{m}} C_{i\mathbf{m}} \Delta^{(\mathbf{m})} \left| a, b, \dots, i, \dots, \mathbf{m} \right. \right\rangle^2 \sim \left\langle \bar{S}_{\mathbf{m}} C_{i\mathbf{m}} \Delta^{(\mathbf{m})} \left| a, b, \dots, [i\mathbf{m}], \dots, i, \dots, \mathbf{m} \right. \right\rangle^2 = \frac{[\alpha_s]}{\epsilon} \left\langle \Gamma_{[i\mathbf{m}], f_{[i\mathbf{m}]} \rightarrow f_i f_{\mathbf{m}}} \left| a, b, \dots, [i\mathbf{m}], \dots \right. \right\rangle^2$$

$$\sum_n \sum_{i \in \mathcal{B}_{N+1,n}(\mathbf{m}_{q\rho})} \delta_{ig} \langle \bar{S}_{\mathbf{m}} C_{i\mathbf{m}} \Delta^{(\mathbf{m})} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m}_{q\rho})] \rangle = \sum_{n'} \sum_{i \in \mathcal{B}_{N,n'}} \delta_{iq\rho} \frac{[\alpha_s]}{\epsilon} \langle \Gamma_{i,q \rightarrow gq} F_{\text{LM}}^{ab}[\mathcal{B}_{N,n'}] \rangle$$



Fully general framework: final formula $pp \rightarrow N \text{ jets} + X @ \text{NNLO}$ can fit a slide

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}}$$

double-virtual
real-virtual
double-real
coll. renor.

$$= d\hat{\sigma}_{ab}^{\text{SU}} + d\hat{\sigma}_{ab}^{\text{DU}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

single-unr.
double-unr.
fully-resolved

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]

fully-resolved = (N+2)-body kinematics
 single-unresolved = (N+1)-body kinematics
 double-unresolved = N-body kinematics

- ❖ Formulas are compact
- ❖ **Very few inputs for actual implementation**
- ❖ **Modular structure**
- ❖ **Easy to adapt to lepton-lepton and lepton-hadron collisions**
- ❖ **EW corrections(?)**

Fully general framework: final formula $pp \rightarrow N\text{jets} + X@NNLO$ can fit a slide

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue, Tresoldi '25]

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = \boxed{d\hat{\sigma}_{ab}^{\text{SU}}} + \boxed{d\hat{\sigma}_{ab}^{\text{DU}}} + \boxed{d\hat{\sigma}_{ab}^{\text{FR}}}$$

double-virtual
real-virtual
double-real
coll. renor.

single-unr.
double-unr.
fully-resolved

fully-resolved = (N+2)-body kinematics
 single-unresolved = (N+1)-body kinematics
 double-unresolved = N-body kinematics

$$d\hat{\sigma}_{ab}^{\text{FR}} = \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \Theta_{mn} \mathbb{F}_{\text{LM}}^{ab, \text{DS}}[\mathbf{m}, \mathbf{n}] \rangle + \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \mathbb{F}_{\text{LM}}^{ab, \text{DS}}[\mathbf{m}, \mathbf{n}] \rangle$$

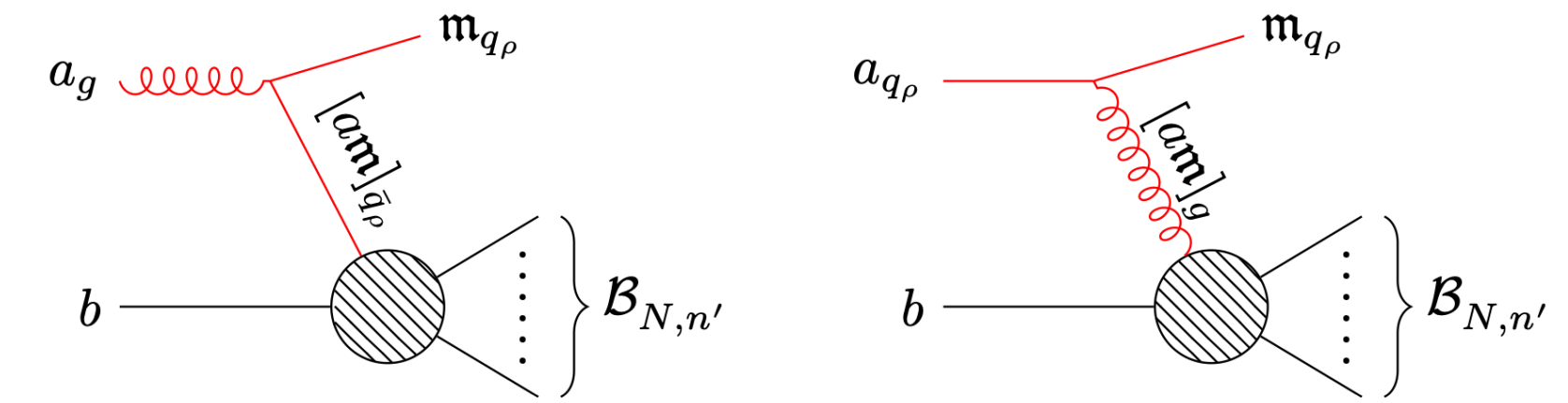
$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{SU, sb, a}} &= [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(a, \mathbf{m})} \omega_{a||\mathbf{n}}^{\mathbf{ma}, \mathbf{na}} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{\mathbf{am}}}{2}\right) \hat{P}_{xa}^{(0)} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\
 &\quad \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right\}, \\
 d\hat{\sigma}_{ab}^{\text{SU, sb, b}} &= [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(b, \mathbf{m})} \omega_{b||\mathbf{n}}^{\mathbf{mb}, \mathbf{nb}} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{\mathbf{bm}}}{2}\right) F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \hat{P}_{xb}^{(0)} \rangle \right. \\
 &\quad \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle \right\}, \\
 d\hat{\sigma}_{ab}^{\text{SU, el}} &= \sum_{\mathbf{m}} \sum_n \left\{ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [\delta_{\mathbf{mg}} I_{\text{T}}^{(0)}(E_{\mathbf{m}}) + \bar{\delta}_{\mathbf{mg}} I_{\text{T}}^{(0)}(E_{\text{max}})] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\
 &\quad - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{i||\mathbf{n}}^{\mathbf{mi}, \mathbf{ni}} \log\left(\frac{\eta_{i\mathbf{m}}}{2}\right) \Delta^{(\mathbf{m})} [\delta_{\mathbf{mg}} (\gamma_i + 2\mathbf{T}_i^2 L_i(E_{\mathbf{m}})) + \bar{\delta}_{\mathbf{mg}} (\gamma_i + 2\mathbf{T}_i^2 L_i)] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\
 &\quad - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{mi}, \mathbf{ni}} \log\left(\frac{\eta_{i\mathbf{m}}}{4(1-\eta_{i\mathbf{m}})}\right) \Delta^{(\mathbf{m})} [\gamma_{\mathbf{m}} + 2\mathbf{T}_q^2 L_{\mathbf{m}} \bar{\delta}_{\mathbf{mg}}] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\
 &\quad + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{RV, fin}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \Big\} + \sum_n \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{mi}, \mathbf{ni}} \Delta^{(\mathbf{m})} [\gamma_g^{\perp, r} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \\
 &\quad + \gamma_g^{\perp} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) F_{\text{LM}, \mu\nu}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \rangle.
 \end{aligned}$$

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{DU, db}} &= \sum_{x, y} \sum_n [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xy}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{yb}^{\text{NLO}} \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU, sb, a}} &= \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes \mathcal{F}^{xb}[\mathcal{B}_{N, n}] \rangle + [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NNLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N, n}] \rangle \right] \\
 &\quad + \sum_{n'} [\alpha_s]^2 \langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n'}]] \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU, sb, b}} &= \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{F}^{ax}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle + [\alpha_s]^2 \langle F_{\text{LM}}^{ax}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{xb}^{\text{NNLO}} \rangle \right] \\
 &\quad + \sum_{n'} [\alpha_s]^2 \langle [\mathcal{W}_b^{b||\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n'}]] \otimes \mathcal{P}_{bb}^{\mathcal{W}} \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU, el}} &= \sum_n \left\{ [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n}] \rangle \right. \\
 &\quad + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \langle [\theta_{\mathcal{H}_f} \gamma_{z, f_i \rightarrow f_i g}^{\mathcal{W}}(L_i) \mathcal{W}_i^{i||\mathbf{n}, \text{fin}} + \delta^{(0)} \mathcal{W}_i^{\mathbf{m}||\mathbf{n}, \text{fin}} + \delta^{\perp, (0)} \mathcal{W}_r^{(i)}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n}] \rangle \\
 &\quad \left. + [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LV, fin}}^{ab}[\mathcal{B}_{N, n}] \rangle + \langle F_{\text{LV}^2, \text{fin}}^{ab}[\mathcal{B}_{N, n}] \rangle + \langle F_{\text{VV, fin}}^{ab}[\mathcal{B}_{N, n}] \rangle \right\}
 \end{aligned}$$

Fully general framework: final formula $pp \rightarrow N\text{jets} + X@NNLO$ can fit a slide

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = d\hat{\sigma}_{ab}^{\text{SU}} + d\hat{\sigma}_{ab}^{\text{DU}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

double-virtual
real-virtual
double-real
coll. renor.
single-unr.
double-unr.
fully-resolved



Very few inputs for actual implementation

- List of N, N+1 and N+2 external particles → select ingredients
- Born-level configurations compatible with the given theory/process
- 1-loop, 2-loop N-point finite remainders → dedicated calculation needed
- details (partitions, azimuthal direction, etc) → optimise for a given process



1. Fix initial-state partons (ab)
2. Sum over all partons x (y) compatible with 1.
3. Sum over all Born-level final-state configurations n compatible with 1. and 2.
→ THIS CAN BE PARALLELISED!
4. Universal ingredients (provided in ancillary files) need to be coded once for all

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{DU,db}} &= \sum_{x,y} \sum_n [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xy}[\mathcal{B}_{N,n}] \otimes \mathcal{P}_{yb}^{\text{NLO}} \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU,sb,a}} &= \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes \mathcal{F}^{xb}[\mathcal{B}_{N,n}] \rangle + [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NNLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N,n}] \rangle \right] \\
 &\quad + \sum_{n'} [\alpha_s]^2 \langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||n,\text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n'}]] \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU,sb,b}} &= \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{F}^{ax}[\mathcal{B}_{N,n}] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle + [\alpha_s]^2 \langle F_{\text{LM}}^{ax}[\mathcal{B}_{N,n}] \otimes \mathcal{P}_{xb}^{\text{NNLO}} \rangle \right] \\
 &\quad + \sum_{n'} [\alpha_s]^2 \langle [\mathcal{W}_b^{b||n,\text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n'}]] \otimes \mathcal{P}_{bb}^{\mathcal{W}} \rangle \\
 d\hat{\sigma}_{ab}^{\text{DU,el}} &= \sum_n \left\{ [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \rangle \right. \\
 &\quad + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \langle [\theta_{\mathcal{H}_f} \gamma_{z,f_i \rightarrow f_{ig}}^{\mathcal{W}}(L_i) \mathcal{W}_i^{i||n,\text{fin}} + \delta^{(0)} \mathcal{W}_i^{m||n,\text{fin}} + \delta^{\perp,(0)} \mathcal{W}_r^{(i)}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \rangle \\
 &\quad \left. + [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LV,fin}}^{ab}[\mathcal{B}_{N,n}] \rangle + \langle F_{\text{LV}^2,\text{fin}}^{ab}[\mathcal{B}_{N,n}] \rangle + \langle F_{\text{VV,fin}}^{ab}[\mathcal{B}_{N,n}] \rangle \right\}
 \end{aligned}$$

Fully general framework: final formula $pp \rightarrow N \text{ jets} + X @ \text{NNLO}$ can fit a slide

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = d\hat{\sigma}_{ab}^{\text{SU}} + d\hat{\sigma}_{ab}^{\text{DU}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

$\vdots$
 double-virtual

 $\vdots$
 real-virtual

 $\vdots$
 double-real

 $\vdots$
 coll. renor.

 $\vdots$
 single-unr.

 $\vdots$
 double-unr.

 $\vdots$
 fully-resolved

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{SU, sb, a}} &= [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(a, \mathbf{m})} \omega_{a||n}^{ma, na} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{am}}{2}\right) \hat{P}_{xa}^{(0)} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\
 &\quad \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right\}, \\
 d\hat{\sigma}_{ab}^{\text{SU, sb, b}} &= [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(b, \mathbf{m})} \omega_{b||n}^{mb, nb} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{bm}}{2}\right) F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \hat{P}_{xb}^{(0)} \rangle \right. \\
 &\quad \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle \right\}, \\
 d\hat{\sigma}_{ab}^{\text{SU, el}} &= \sum_{\mathbf{m}} \sum_n \left\{ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\mathbf{m}}) + \bar{\delta}_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\text{max}})] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\
 &\quad - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{i||n}^{mi, ni} \log\left(\frac{\eta_{im}}{2}\right) \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} (\gamma_i + 2\mathbf{T}_i^2 L_i(E_{\mathbf{m}})) + \bar{\delta}_{\mathbf{m}g} (\gamma_i + 2\mathbf{T}_i^2 L_i)] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\
 &\quad - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||n}^{mi, ni} \log\left(\frac{\eta_{im}}{4(1 - \eta_{im})}\right) \Delta^{(\mathbf{m})} [\gamma_{\mathbf{m}} + 2\mathbf{T}_q^2 L_{\mathbf{m}} \bar{\delta}_{\mathbf{m}g}] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\
 &\quad + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{RV, fin}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \Big\} + \sum_n \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||n}^{mi, ni} \Delta^{(\mathbf{m})} [\gamma_g^{\perp, r} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \\
 &\quad + \gamma_g^{\perp} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) F_{\text{LM}, \mu\nu}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \rangle.
 \end{aligned}$$

Very few inputs for actual implementation

- Same inputs as before
 - + spin correlated matrix elements
 - + (N+1) configurations compatible with the given theory/process
- 1-loop (N+1)-point finite remainders → dedicated calculation needed
- details (partition functions under limits, azimuthal direction, damping factors, etc) → easy to optimise for a given process

Fully general framework: final formula $pp \rightarrow N\text{jets} + X@NNLO$ can fit a slide

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = \boxed{d\hat{\sigma}_{ab}^{\text{SU}}} + \boxed{d\hat{\sigma}_{ab}^{\text{DU}}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

double-virtual
real-virtual
double-real
coll. renor.

single-unr.
double-unr.
fully-resolved

Modular structure:

- **boosted contributions** (two copies of the same structure + double boosted)
- **Elastic contributions**

$$d\hat{\sigma}_{ab}^{\text{SU, sb, a}} = [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(a, \mathbf{m})} \omega_{a||\mathbf{n}}^{ma, na} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{am}}{2}\right) \hat{P}_{xa}^{(0)} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\ \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right\},$$

$$d\hat{\sigma}_{ab}^{\text{SU, sb, b}} = [\alpha_s] \sum_x \sum_{\mathbf{m}} \sum_n \left\{ \langle \mathcal{O}_{\text{NLO}}^{(b, \mathbf{m})} \omega_{b||\mathbf{n}}^{mb, nb} \Delta^{(\mathbf{m})} \log\left(\frac{\eta_{bm}}{2}\right) F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \hat{P}_{xb}^{(0)} \rangle \right. \\ \left. + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{LM}}^{ax}[\mathcal{B}_{N+1, n}(\mathbf{m})] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle \right\}.$$

$$d\hat{\sigma}_{ab}^{\text{SU, el}} = \sum_{\mathbf{m}} \sum_n \left\{ [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\mathbf{m}}) + \bar{\delta}_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\text{max}})] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \right. \\ - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{i||\mathbf{n}}^{mi, ni} \log\left(\frac{\eta_{im}}{2}\right) \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} (\gamma_i + 2T_i^2 L_i(E_{\mathbf{m}})) + \bar{\delta}_{\mathbf{m}g} (\gamma_i + 2T_i^2 L_i)] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\ - \sum_{i \in \mathcal{H}} [\alpha_s] \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{mi, ni} \log\left(\frac{\eta_{im}}{4(1-\eta_{im})}\right) \Delta^{(\mathbf{m})} [\gamma_{\mathbf{m}} + 2T_q^2 L_{\mathbf{m}} \bar{\delta}_{\mathbf{m}g}] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \\ + \langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{RV, fin}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m})] \rangle \Big\} + \sum_n \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \langle \mathcal{O}_{\text{NLO}}^{(i, \mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{mi, ni} \Delta^{(\mathbf{m})} [\gamma_g^{\perp, r} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \\ + \gamma_g^{\perp} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) F_{\text{LM, } \mu\nu}^{ab}[\mathcal{B}_{N+1, n}(\mathbf{m}_g)] \rangle \Big\}.$$

$$d\hat{\sigma}_{ab}^{\text{DU, db}} = \sum_{x, y} \sum_n [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes F_{\text{LM}}^{xy}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{yb}^{\text{NLO}} \rangle$$

$$d\hat{\sigma}_{ab}^{\text{DU, sb, a}} = \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{P}_{xa}^{\text{NLO}} \otimes \mathcal{F}^{xb}[\mathcal{B}_{N, n}] \rangle + [\alpha_s]^2 \langle \mathcal{P}_{xa}^{\text{NNLO}} \otimes F_{\text{LM}}^{xb}[\mathcal{B}_{N, n}] \rangle \right] \\ + \sum_{n'} [\alpha_s]^2 \langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes [\mathcal{W}_a^{a||\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n'}]] \rangle$$

$$d\hat{\sigma}_{ab}^{\text{DU, sb, b}} = \sum_x \sum_n \left[[\alpha_s] \langle \mathcal{F}^{ax}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{xb}^{\text{NLO}} \rangle + [\alpha_s]^2 \langle F_{\text{LM}}^{ax}[\mathcal{B}_{N, n}] \otimes \mathcal{P}_{xb}^{\text{NNLO}} \rangle \right] \\ + \sum_{n'} [\alpha_s]^2 \langle [\mathcal{W}_b^{b||\mathbf{n}, \text{fin}} \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n'}]] \otimes \mathcal{P}_{bb}^{\mathcal{W}} \rangle$$

$$d\hat{\sigma}_{ab}^{\text{DU, el}} = \sum_n \left\{ [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n}] \rangle \right. \\ + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \langle [\theta_{\mathcal{H}_f} \gamma_{z, f_i \rightarrow f_i g}^{\mathcal{W}}(L_i) \mathcal{W}_i^{i||\mathbf{n}, \text{fin}} + \delta^{(0)} \mathcal{W}_i^{\mathbf{m}||\mathbf{n}, \text{fin}} + \delta^{\perp, (0)} \mathcal{W}_r^{(i)}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N, n}] \rangle \\ \left. + [\alpha_s] \langle I_{\text{T}}^{(0)} \cdot F_{\text{LV, fin}}^{ab}[\mathcal{B}_{N, n}] \rangle + \langle F_{\text{LV}^2, \text{fin}}^{ab}[\mathcal{B}_{N, n}] \rangle + \langle F_{\text{VV, fin}}^{ab}[\mathcal{B}_{N, n}] \rangle \right\}$$

Fully general framework: final formula $pp \rightarrow N\text{jets} + X@NNLO$ can fit a slide

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = d\hat{\sigma}_{ab}^{\text{SU}} + d\hat{\sigma}_{ab}^{\text{DU}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

$\vdots$
 double-virtual

 $\vdots$
 real-virtual

 $\vdots$
 double-real

 $\vdots$
 coll. renor.

$\vdots$
 single-unr.

 $\vdots$
 double-unr.

 $\vdots$
 fully-resolved

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{SU,el}} = & \sum_{\mathbf{m}} \sum_n \left\{ [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} [\delta_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\mathbf{m}}) + \bar{\delta}_{\mathbf{m}g} I_{\text{T}}^{(0)}(E_{\text{max}})] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \right. \\
 & - \sum_{i \in \mathcal{H}} [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{i||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \log\left(\frac{\eta_{i\mathbf{m}}}{2}\right) \Delta^{(\mathbf{m})} \left[\delta_{\mathbf{m}g} \left(\gamma_i + 2\mathbf{T}_i^2 L_i(E_{\mathbf{m}}) \right) + \bar{\delta}_{\mathbf{m}g} \left(\gamma_i + 2\mathbf{T}_i^2 L_i \right) \right] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \\
 & - \sum_{i \in \mathcal{H}} [\alpha_s] \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \log\left(\frac{\eta_{i\mathbf{m}}}{4(1-\eta_{i\mathbf{m}})}\right) \Delta^{(\mathbf{m})} \left[\gamma_{\mathbf{m}} + 2\mathbf{T}_q^2 L_{\mathbf{m}} \bar{\delta}_{\mathbf{m}g} \right] F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \\
 & + \left\langle \mathcal{O}_{\text{NLO}}^{(\mathbf{m})} \Delta^{(\mathbf{m})} F_{\text{RV,fin}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m})] \right\rangle \left. \right\} + \sum_n \sum_{i \in \mathcal{H}} \frac{[\alpha_s]}{2} \left\langle \mathcal{O}_{\text{NLO}}^{(i,\mathbf{m})} \omega_{\mathbf{m}||\mathbf{n}}^{\mathbf{m}i,\mathbf{n}i} \Delta^{(\mathbf{m})} \left[\gamma_g^{\perp,\text{r}} F_{\text{LM}}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m}_g)] \right. \right. \\
 & \left. \left. + \gamma_g^{\perp} (r_i^{\mu} r_i^{\nu} + g^{\mu\nu}) F_{\text{LM},\mu\nu}^{ab}[\mathcal{B}_{N+1,n}(\mathbf{m}_g)] \right] \right\rangle.
 \end{aligned}$$

Lepton-lepton colliders:

- Final formulas are written as expressions that are applied to each of the resolved partons
- Given the modular structure one can easily remove the initial-state contributions (either partially, e.g. for DIS, or entirely)

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{DU,el}} = & \sum_n \left\{ [\alpha_s]^2 \left\langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \right\rangle \right. \\
 & + [\alpha_s]^2 \sum_{i \in \mathcal{H}} \left\langle \left[\theta_{\mathcal{H}_f} \gamma_{z,f_i \rightarrow f_{ig}}^{\mathcal{W}}(L_i) \mathcal{W}_i^{i||\mathbf{n},\text{fin}} + \delta^{(0)} \mathcal{W}_i^{\mathbf{m}||\mathbf{n},\text{fin}} + \delta^{\perp,(0)} \mathcal{W}_r^{(i)} \right] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \right\rangle \\
 & \left. + [\alpha_s] \left\langle I_{\text{T}}^{(0)} \cdot F_{\text{LV,fin}}^{ab}[\mathcal{B}_{N,n}] \right\rangle + \left\langle F_{\text{LV}^2,\text{fin}}^{ab}[\mathcal{B}_{N,n}] \right\rangle + \left\langle F_{\text{VV,fin}}^{ab}[\mathcal{B}_{N,n}] \right\rangle \right\}
 \end{aligned}$$

Fully general framework: final formula $pp \rightarrow N\text{jets} + X@NNLO$ can fit a slide

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} = d\hat{\sigma}_{ab}^{\text{SU}} + d\hat{\sigma}_{ab}^{\text{DU}} + d\hat{\sigma}_{ab}^{\text{FR}}$$

$\vdots$
 double-virtual

 $\vdots$
 real-virtual

 $\vdots$
 double-real

 $\vdots$
 coll. renor.

 $\vdots$
 single-unr.

 $\vdots$
 double-unr.

 $\vdots$
 fully-resolved

$$\Omega_1 = \sum_{(ij)} \bar{C}_{im} \bar{C}_{jn} [dp_m][dp_n] \omega^{mi,nj} + \sum_i \left[\bar{C}_{in} \theta^{(a)} + \bar{C}_{mn} \theta^{(b)} + \bar{C}_{im} \theta^{(c)} + \bar{C}_{mn} \theta^{(d)} \right] [dp_m][dp_n] \bar{C}_{mn,i} \omega^{mi,ni}$$

$$d\hat{\sigma}_{ab}^{\text{FR}} = \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \Theta_{mn} \mathbb{F}_{\text{LM}}^{ab, \text{DS}}[\mathbf{m}, \mathbf{n}] \rangle + \langle \bar{S}_{mn} \bar{S}_n \Omega_1 \Delta^{(mn)} \mathbb{F}_{\text{LM}}^{ab, \text{DS}}[\mathbf{m}, \mathbf{n}] \rangle$$

DS = double-soft singularities

1. Fully regulated and can be computed numerically
2. Sector functions $\theta^{(a, \dots, d)}$ are “absorbed in the parametrisation”, so no sharp step functions in the code
3. Each sector is treated separately, using different parametrisation [\[Czakon '10+'11\]](#)
→ THIS CAN BE PARALLELISED!
4. **In each sector and partition**, one needs to consider at most **7 different kinematic configurations**, each resulting from applying one or more of the operators to the double-real emission kinematics
5. Sectors count:
 - $N + 2$ **triple-collinear** partitions: **4 subsectors** each
 - $(N + 2)(N + 1)$ **double-collinear** partitions



Universal building blocks

$$\begin{aligned}
 d\hat{\sigma}_{ab}^{\text{NNLO}} &= d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} \\
 &\quad \begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ \text{double-virtual} & \text{real-virtual} & \text{double-real} & \text{coll. renor.} \end{array} \\
 &\Rightarrow d\hat{\sigma}_{ab}^{\text{DU}} \Rightarrow d\hat{\sigma}_{ab}^{\text{DU,el}} = \sum_n \left\{ [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \rangle + \dots \right. \\
 &\quad \begin{array}{ll} I_{\text{cc}}^{\text{fin}} & = \text{double colour correlations } T_i \cdot T_j, (T_i \cdot T_j) \cdot (T_k \cdot T_l) \\ I_{\text{ss}}^{\text{fin}} & = \text{subset of colour correlations of soft origin } T_i \cdot T_j \\ I_{\text{tri}}^{\text{fin}} & = \text{triple colour correlations } T_i \cdot T_j \cdot T_k \\ I_{\text{unc}}^{\text{fin}} & = \text{colour un-correlations} \end{array}
 \end{aligned}$$

The idea is to subdivide integrated counterterms according to their **kinematical structure** AND according to their **color structure**

Universal building blocks: example

$$d\hat{\sigma}_{ab}^{\text{NNLO}} = d\hat{\sigma}_{ab}^{\text{VV}} + d\hat{\sigma}_{ab}^{\text{RV}} + d\hat{\sigma}_{ab}^{\text{RR}} + d\hat{\sigma}_{ab}^{\text{PDFs}} \quad \Rightarrow \quad d\hat{\sigma}_{ab}^{\text{DU}} \quad \Rightarrow \quad d\hat{\sigma}_{ab}^{\text{DU,el}} = \sum_n \left\{ [\alpha_s]^2 \langle [I_{\text{cc}}^{\text{fin}} + I_{\text{ss}}^{\text{fin}} + I_{\text{tri}}^{\text{fin}} + I_{\text{unc}}^{\text{fin}}] \cdot F_{\text{LM}}^{ab}[\mathcal{B}_{N,n}] \rangle + \dots \right.$$

double-virtual

real-virtual

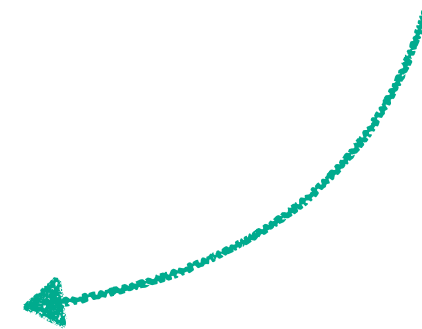
double-real

coll. renor.

- $I_{\text{cc}}^{\text{fin}}$ = double colour correlations $T_i \cdot T_j, (T_i \cdot T_j) \cdot (T_k \cdot T_l)$
- $I_{\text{ss}}^{\text{fin}}$ = subset of colour correlations of soft origin $T_i \cdot T_j$
- $I_{\text{tri}}^{\text{fin}}$ = triple colour correlations $T_i \cdot T_j \cdot T_k$
- $I_{\text{unc}}^{\text{fin}}$ = colour un-correlations

The idea is to subdivide integrated counterterms according to their **kinematical structure** AND according to their **color structure**

$$I_{\text{cc}}^{\text{fin}} = \frac{1}{2} (I_{\text{T}}^{(0)})^2 + K I_{\text{T}}^{(0)} + \dots$$



Example of **iterated NLO structure**. Also, it results from the combination of terms arising from (almost) every bit of $d\hat{\sigma}_{ab}^{\text{NNLO}}$

Quick reminder from NLO: simple interplay between $\underbrace{[V + S_i R + (I - S_i) C_{ij} R]_{\text{elastic}}}_{I_T(\epsilon) = I_V(\epsilon) + I_S(\epsilon) + I_C(\epsilon)}$ and $\underbrace{[(1 - S_i) C_{ij} R]_{\text{boost}} + \text{PDFs}}_{\langle \mathcal{P}_{aa}^{\text{NLO}} \otimes F_{\text{LM}} \rangle + \langle F_{\text{LM}} \otimes \mathcal{P}_{bb}^{\text{NLO}} \rangle}$

Moving forward to NNLO

Similar guiding principle @NNLO:

starting from **IR poles of double-virtual** [Catani '98], find **subtraction terms** that can “**complete**” it: identify structures similar to those encountered at NLO → write $\text{NNLO} \sim \text{NLO}^2$ as much as possible

❖ Rewrite the VV in term of NLO operators:

$$\begin{aligned} \langle F_{\text{VV}} \rangle &= [\alpha_s]^2 \left\langle \left[\frac{1}{2} I_V^2(\epsilon) - \frac{\Gamma(1-\epsilon)}{e^{\epsilon\gamma_E}} \left(\frac{\beta_0}{\epsilon} I_V(\epsilon) - \left(\frac{\beta_0}{\epsilon} + K \right) I_V(2\epsilon) \right) \right] \cdot F_{\text{LM}} \right\rangle \\ &+ [\alpha_s]^2 \left\langle \left[-\frac{1}{2} [\bar{I}_1(\epsilon), \bar{I}_1^\dagger(\epsilon)] + \mathcal{H}_{2,\text{tc}} + \mathcal{H}_{2,\text{tc}}^\dagger + \mathcal{H}_{2,\text{cd}} + \mathcal{H}_{2,\text{cd}}^\dagger \right] \cdot F_{\text{LM}} \right\rangle \\ &+ [\alpha_s] \langle I_V(\epsilon) \cdot F_{\text{LV}}^{\text{fin}} \rangle + \langle F_{\text{LV}^2}^{\text{fin}} \rangle + \langle F_{\text{VV}}^{\text{fin}} \rangle . \end{aligned}$$

❖ Identify features that can guide you

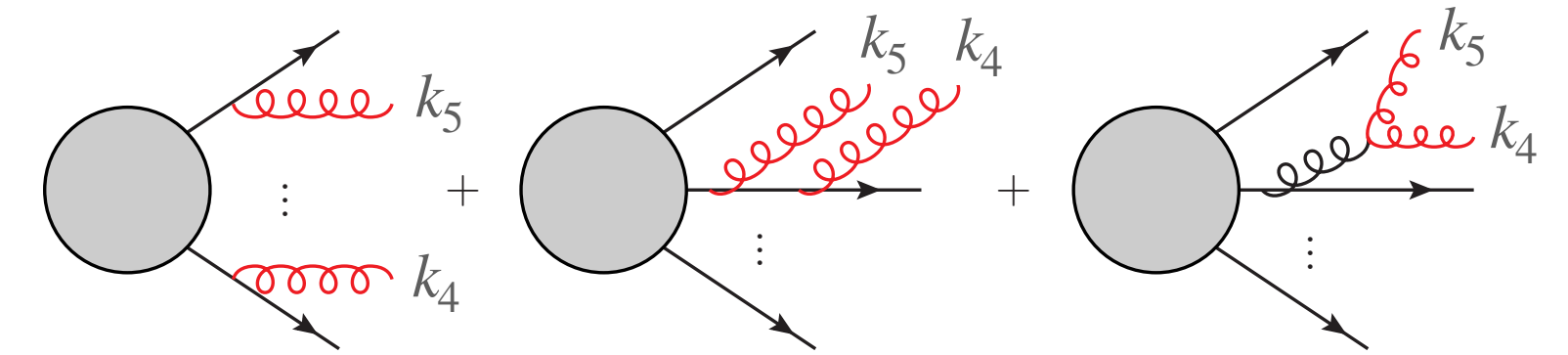
- different **powers/arguments/prefactors**
- different type of **color-correlations**

$$\left\{ \begin{array}{l} T_i \cdot T_j \\ T_i \cdot T_j \cdot T_k \\ (T_i \cdot T_j) \cdot (T_k \cdot T_l) \end{array} \right.$$

→ **specific pattern of cancellation.**

Follow the (colored) crumbs

- ❖ Search for similar features among the various subtraction terms.
For instance double and quartic color correlations:



Double soft

[Catani, Grazzini '99]

Factorised term

$$(T_i \cdot T_j) \cdot (T_k \cdot T_l)$$

$\Rightarrow$

$$\langle S_{mn} \Theta_{mn} F_{LM}(\mathbf{m}, \mathbf{n}) \rangle_{T^4} = [\alpha_s]^2 \frac{1}{2} \langle I_S^2(\epsilon) \rangle \cdot F_{LM}$$

- ❖ Combine terms such that manifest cancellations occur without any process-aware manipulation

$I_S^2(\epsilon) + I_V^2(\epsilon)$ free of quartic color-correlated poles

Iterations of NLO $\rightarrow$ almost $(I_S + I_V)^2 \rightarrow$ to obtain a NLO² object we need the product

- ❖ The “missing” products we look at the limits of the real-virtual contribution. The integrated soft subtraction term can be almost fully written in terms of NLO-like operators

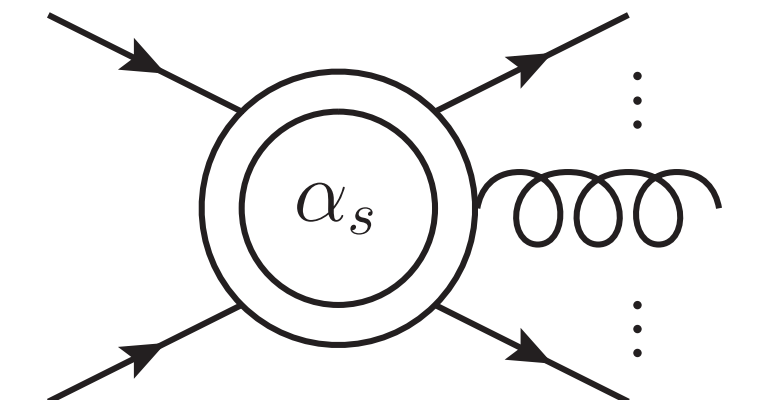
Soft real-virtual

[Catani, Grazzini '00]

$$\langle S_m F_{RV}(\mathbf{m}) \rangle = [\alpha_s]^2 \left\langle \frac{1}{2} [I_S(\epsilon) \cdot I_V(\epsilon) + I_V(\epsilon) \cdot I_S(\epsilon)] \right\rangle \cdot F_{LM} + \dots$$

Triple-color correlations:

- Vanish for $N_p \geq 4$
- Finite after integration for FSR

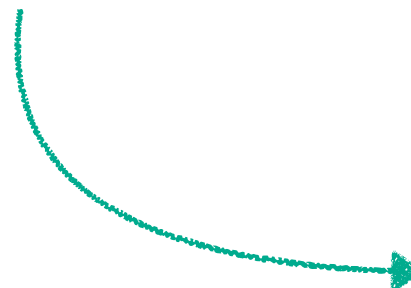


The pie so far

$\langle F_{LVV} \rangle$	$\frac{1}{2} I_V^2(\epsilon)$	} $= I_T^2 - I_C^2$ no singular c.c.
$\langle S_{mn} \Theta_{mn} F_{LM}(\mathfrak{m}, \mathfrak{n}) \rangle$	$\frac{1}{2} I_S^2(\epsilon)$	
$\langle S_{\mathfrak{m}} F_{RV}(\mathfrak{m}) \rangle$	$\frac{1}{2} [I_S(\epsilon) \cdot I_V(\epsilon) + I_V(\epsilon) \cdot I_S(\epsilon)]$	



Almost reconstruct $I_T^2(\epsilon)$
 $\rightarrow$ look at collinear

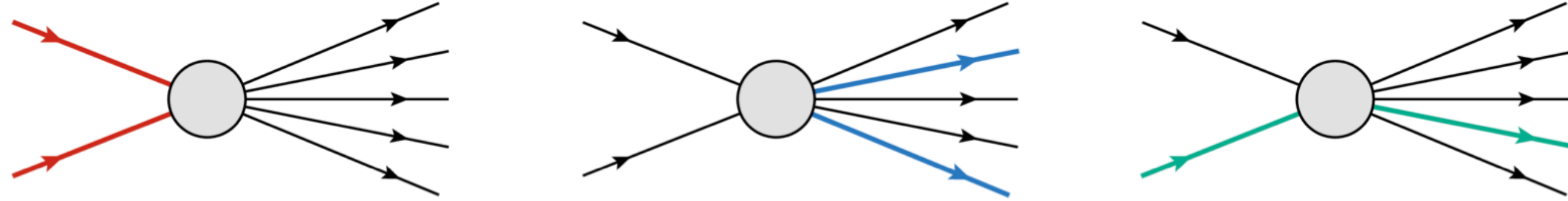


combine sectors at a symbolic level before integrating

$$I_T(\epsilon) = I_V(\epsilon) + I_S(\epsilon) + I_C(\epsilon) \text{ finite}$$

Pictorial conclusions

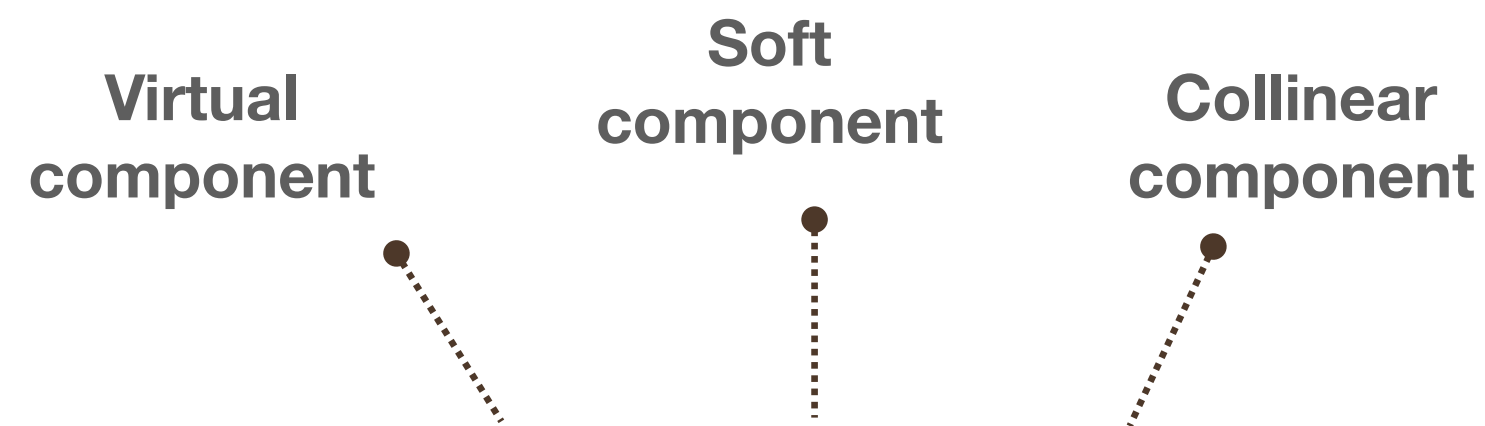
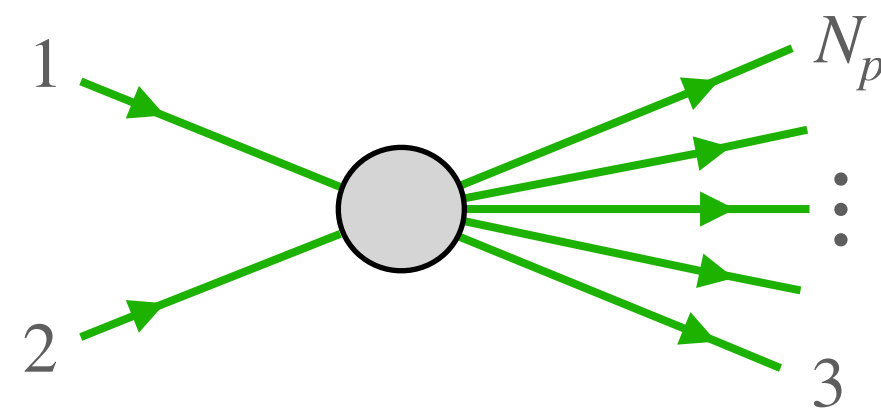
Previous studies:



Can we generalise? yes!

[Devoto, Melnikov, Röntsch, [CSS](#), Tagliabue + Tresoldi '24, '25]

❖ Introduce **NLO-like universal operators** that describe **virtual, soft and collinear singularities**, and combine into **finite quantities**



$$I_T(\epsilon) = I_V(\epsilon) + I_S(\epsilon) + I_C(\epsilon)$$

Free of poles
Fully general in the number of partons

❖ Reduce **NNLO** corrections to **iterations of these operators** → demonstrate **cancellations prior to explicit evaluation**

$$d\hat{\sigma}^{\text{NNLO}} = \underbrace{d\hat{\sigma}^{\text{VV}}}_{\text{Double-Virtual}} + \underbrace{d\hat{\sigma}^{\text{RV}}}_{\text{Real-Virtual}} + \underbrace{d\hat{\sigma}^{\text{RR}}}_{\text{Double-Real}} + \underbrace{d\hat{\sigma}^{\text{pdf}}}_{\text{PDFs Renor.}}$$

$$= \frac{[\alpha_s]^2}{2} \langle M_0 | [I_V + I_S + I_C]^2 | M_0 \rangle + \dots \equiv \langle M_0 | I_T^2 | M_0 \rangle + \dots$$