Exercise session 1 - solutions

1. Laser propagation in plasma

A laser can propagate in plasma if its angular frequency ω_ℓ is greater than the plasma angular frequency ω_p:

$$\omega_\ell > \omega_p$$

The laser angular frequency is:

$$\omega_\ell = \frac{2\pi c}{\lambda}$$

The plasma frequency is given by:

$$\omega_p = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}$$

Laser frequencies:

$$\omega_{\ell_1} = 2.36 \times 10^{15} \text{ rad/s}$$
$$\omega_{\ell_2} = 1.88 \times 10^{14} \text{ rad/s}$$

Plasma frequencies:

$$\omega_{p1} = 1.78 \times 10^{13} \text{ rad/s}$$
$$\omega_{p2} = 3.99 \times 10^{14} \text{ rad/s}$$

Comparison and Result:

- (a) + (c): $\omega_1 > \omega_{p1} \Rightarrow$ **Yes, propagation is possible**
- (a) + (d): $\omega_1 > \omega_{p2} \Rightarrow$ **Yes, propagation is possible**
- (b) + (c): $\omega_2 > \omega_{p1} \Rightarrow$ **Yes, propagation is possible**
- (b) + (d): $\omega_2 < \omega_{p2} \Rightarrow$ **No, propagation is not possible**

2. Wakefield excitation length

1. (a) Since $n_b > n_{pe}$ the regime is non-linear:

$$E_{acc} \sim \sqrt{\frac{n_b}{n_{pe}}} E_{WB}$$

A transformer ratio equal to 1 means that $E_{acc} = E_{decc}$, calculating it with the values given:

$$E_{decc} \sim 21\, GV/m.$$

In principle, the bunch stops driving wakefields once its energy is depleted, such that the length over which it drives wakefields is given by:

$$\frac{E_{particles}}{E_{decc}} = \frac{10}{21} \sim 0.47\,m$$

2. (b) We note that the depletion length in the case of a laser pulse can be expressed in two different ways, depending whether the regime is linear, or non-linear:

$$\text{Linear} : a_0^2 \ll 1 : \quad L_{\text{pd}} \sim \frac{1}{a_0^2}\left(\frac{\omega_\ell}{\omega_{pe}}\right)^2 \lambda_{pe}$$

$$\text{Non-linear} : a_0^2 \gg 1 : \quad L_{\text{pd}} \sim \frac{a_0}{\pi}\left(\frac{\omega_\ell}{\omega_{pe}}\right)^2 \lambda_{pe}$$

First, we calculate a_0 for the parameters given:

$$a_0 \sim \sqrt{\frac{I\,[\text{W/cm}^2]}{1.37 \times 10^{18}}}\,\lambda_\ell\,[\mu\text{m}] = 1.52$$

$$a_0^2 = 2.32 >> 1$$

We will therefore use the depletion length formula in the non-linear case. We calculate $\omega_\ell = 2.35 \times 10^{15}$ rad/s and $\omega_{pe} = 5.64 \times 10^{13}$ rad/s

$$L_{pd} = 2.84\,cm$$

3. Group velocity and witness electron energy gain

(a) The group velocity of a laser pulse in plasma is given by

$$v_g = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \qquad \omega_p = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}, \qquad \omega = \frac{2\pi c}{\lambda}.$$

If $\omega_p \geq \omega$, the laser cannot propagate (cutoff condition).

(a,c) For $\lambda_1 = 800\,\text{nm}$ and $n_{pe1} = 2 \times 10^{17}\,\text{cm}^{-3}$:

$$v_g = c\left(1 - 5.74 \times 10^{-5}\right)$$

(a,d) For $\lambda_1 = 800\,\text{nm}$ and $n_{pe2} = 3 \times 10^{19}\,\text{cm}^{-3}$:

$$v_g = c\left(1 - 8.65 \times 10^{-3}\right)$$

(b,c) For $\lambda_2 = 10\,\mu$m and $n_{pe1} = 2 \times 10^{17}\,\mathrm{cm}^{-3}$:

$$v_g = c\left(1 - 9.01 \times 10^{-3}\right)$$

(b,d) For $\lambda_2 = 10\,\mu$m and $n_{pe2} = 3 \times 10^{19}\,\mathrm{cm}^{-3}$:

$$\omega_p > \omega \quad \Rightarrow \quad \text{Laser is cut off (no propagation).}$$

(b) The largest energy gain occurs when the accelerating distance times the accelerating gradient is the largest. If we neglect the diffraction of the laser, the accelerating distance is limited by either depletion of the driver, or dephasing of the witness. In the case of the linear regime, $a_0^2 << 1$, these lengths can be expressed as the following:

$$L_{pd} \sim \frac{1}{a_0^2}\,(\frac{\omega_l}{\omega_{pe}})^2\,\lambda_{pe}$$

$$L_d \sim \frac{1}{2}(\frac{\omega_l}{\omega_{pe}})^2\,\lambda_{pe}$$

These two expressions contain ω_{pe} and λ_{pe}, which can both be expressed as a function of n_{pe}. Same for ω_l which can be expressed as a function of λ_l.

We find (for a fixed a_0):

$$L_{pd} \propto L_d \propto \frac{1}{\lambda_l^2\,n_{pe}^{3/2}}$$

The accelerating gradient is:

$$E_{acc} \sim a_0^2\,E_{WB}$$

i.e.,

$$E_{acc} \propto E_{WB} \propto n_{pe}^{1/2}$$

The energy of the witness therefore scales as

$$E_{witness} \propto \frac{1}{\lambda_l^2\,n_{pe}^{3/2}}\,n_{pe}^{1/2} \propto \frac{1}{\lambda_l^2\,n_{pe}}$$

The combination of n_{pe} and λ_l result in the largest energy gain is therefore the smallest n_{pe} and λ_l.

Note that, while in this exercise, a_0 is fixed, a_0 also has a dependency on λ_l.

4. Rayleigh range and self-focusing power

Reyleigh length and self-focusing power threshold formulas:

$$z_R = \frac{\pi w_0^2}{\lambda}, \qquad P_{\text{crit}} = 17\left(\frac{\omega}{\omega_p}\right)^2 \text{ GW}$$

with $\omega = 2\pi c/\lambda$ and $\omega_p = \sqrt{n_e e^2/(\varepsilon_0 m_e)}$.

Solutions:

(a) Rayleigh ranges for $w_0 = 50\ \mu$m:

$$z_R(\lambda_1 = 800\,\text{nm}) = \frac{\pi(50\times10^{-6})^2}{800\times10^{-9}} = 9.8175\times10^{-3}\text{ m} = 9.82 \text{ mm},$$

$$z_R(\lambda_2 = 10\,\mu\text{m}) = \frac{\pi(50\times10^{-6})^2}{10\times10^{-6}} = 7.85398\times10^{-4}\text{ m} = 0.7854 \text{ mm}.$$

(b) Minimum (critical) power for relativistic self-focusing $P_{\text{crit}} = 17(\omega/\omega_p)^2$ GW. Numerical values for the two wavelengths and two densities $n_{pe1} = 2\times10^{17}\text{ cm}^{-3}$, $n_{pe2} = 5\times10^{18}\text{ cm}^{-3}$ (converted to m^{-3}):

$$\begin{aligned}
&\lambda_1 = 800 \text{ nm}:\\
&\quad n_{pe1} = 2\times10^{17}\text{ cm}^{-3}: \quad P_{\text{crit}} \approx 1.481\times10^{14}\text{ W} \approx 1.481\times10^{5}\text{ GW} \approx 148.1\text{ TW},\\
&\quad n_{pe2} = 5\times10^{18}\text{ cm}^{-3}: \quad P_{\text{crit}} \approx 5.938\times10^{12}\text{ W} \approx 5.9\times10^{3}\text{ GW} \approx 5.9\text{ TW};\\
&\lambda_2 = 10\ \mu\text{m}:\\
&\quad n_{pe1} = 2\times10^{17}\text{ cm}^{-3}: \quad P_{\text{crit}} \approx 9.4763\times10^{11}\text{ W} \approx 948\text{ GW}\\
&\quad n_{pe2} = 5\times10^{18}\text{ cm}^{-3}: \quad P_{\text{crit}} \approx 3.7905\times10^{10}\text{ W} \approx 38\text{ GW}
\end{aligned}$$

5. Dephasing length in PWFA

The drive bunch and the witness propagate with constant group velocity velocities which can be expressed by:

$$v_{b,d} = c\sqrt{1-\frac{1}{\gamma_d^2}}, \qquad v_{b,w} = c\sqrt{1-\frac{1}{\gamma_w^2}}.$$

The witness dephases from the focusing accelerating region of the wakefields when the relative slip equals one quarter of the plasma wavelength:

$$(v_{b,w} - v_{b,d})\, t_{\text{d}} \sim \frac{\lambda_{pe}}{4}.$$

Hence the dephasing length (distance travelled by the witness during t_d) is

$$L_\mathrm{d} \approx c\,t_\mathrm{d} \approx \frac{c\,\lambda_{pe}}{4\,(v_{b,w} - v_{b,d})}.$$

Writing the velocities explicitly in terms of the Lorentz factors gives the exact form

$$\boxed{L_\mathrm{d} = \frac{c\,\lambda_{pe}}{4}\,\frac{1}{\sqrt{1-\frac{1}{\gamma_w^2}} - \sqrt{1-\frac{1}{\gamma_d^2}}}}$$

For highly relativistic beams ($\gamma_d, \gamma_w \gg 1$) one may expand

$$v_b \simeq c\Big(1 - \frac{1}{2\gamma^2}\Big),$$

so that to leading order

$$v_{b,w} - v_{b,d} \simeq \frac{c}{2}\left(\frac{1}{\gamma_d^2} - \frac{1}{\gamma_w^2}\right) = \frac{c}{2}\,\frac{\gamma_w^2 - \gamma_d^2}{\gamma_d^2\gamma_w^2}.$$

$$\boxed{L_\mathrm{d} \simeq \frac{\lambda_{pe}}{2}\,\frac{\gamma_d^2\gamma_w^2}{\gamma_w^2 - \gamma_d^2}}$$

6. Dispersion relation in vacuum

Starting from Maxwell's equations in vacuum:

$$\nabla\cdot\mathbf{E} = 0, \qquad \nabla\cdot\mathbf{B} = 0, \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

Take the curl of Faraday's law and substitute Ampère's law:

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

Using the vector identity $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$ and $\nabla\cdot\mathbf{E} = 0$ yields the wave equation

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0.$$

Assume a plane-wave solution

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp\big[i(\mathbf{k}\cdot\mathbf{r} - \omega t)\big].$$

Substituting into the wave equation gives

$$(-k^2 + \frac{\omega^2}{c^2})\mathbf{E}_0 = 0,$$

which requires

$$k^2 = \frac{\omega^2}{c^2}.$$

Taking the positive-frequency branch yields the vacuum dispersion relation

$$\boxed{\omega = c\,k}$$

(and equivalently $k = \omega/c$), as was to be shown.

1 Exercise session 2 - solutions

1. Efficiency

First, we calculate the power required for 1nC at 1 TeV with a repetition rate of 1kHz.

The number of electrons in 1nC can be calculated as follows:

$$N_e \approx \frac{1 \times 10^{-9}}{1.6 \times 10^{-19}} \approx 6.24 \times 10^9 \,\text{electrons}$$

If each electron acquires 1 TeV of energy, the energy per accelerated bunch is:

$$E_{bunch} \approx N_e\, E_e \approx 6.24 \times 10^9 \cdot 1 \times 10^{12} \cdot 1.6 \times 10^{-19} \approx 1\,\text{kJ}$$

The power required is then simply the energy per bunch multiplied by the repetition rate:

$$P = E_{bunch} \cdot 1\,\text{kHz}$$

So now that we know how much power we would need for these beams to exist with this amount of charge, energy and at this repetition rate we can go back to the initial discussion. The wall-plug to witness efficiency (η_{wp2w})is expressed by:

$$\eta_{wp2w} = \eta_{wp2d}\, \eta_{d2w}\, \eta_{wk2w}$$

where η_{wp2d} is the wall-plug to driver efficiency, η_{d2w} is the driver to witness efficiency and η_{wk2w} is the wake-to-witness efficiency.

We want to use at most 2.7 MW of power, and the power contained in the electron bunches produced will be 1 MW.

In other words, we want:

$$P \cdot \eta_{wp2w} = 2.7\text{MW} \cdot \eta_{wp2w} > 1\text{MW}$$

i.e.,

$$2.7\text{MW} \cdot \eta_{wp2d}\, \eta_{d2w}\, \eta_{wk2w} > 1\text{MW}$$

We solve for η_{wk2w}:

$$\eta_{wk2w} > \frac{1}{2.7\,\eta_{wp2d}\,\eta_{d2w}} = \frac{1}{2.7 \cdot 0.5 \cdot 0.9} \approx 82\%$$

2. Staging

We consider a beam propagating through a vacuum gap with a thin focusing quadrupole. Let the first drift have length d_1 and the second drift d_2. The transport matrix is:

$$T = D(d_2)\,L(f)\,D(d_1),$$

where

$$D(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad L(f) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Step 1: Obtain the transport matrix
First, multiply $L(f)$ by $D(d_1)$:

$$L(f)D(d_1) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & d_1 \\ -1/f & 1 - d_1/f \end{pmatrix}$$

Next, multiply by $D(d_2)$ on the left:

$$T = D(d_2)\,L(f)D(d_1) = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ -1/f & 1 - d_1/f \end{pmatrix} = \begin{pmatrix} 1 - \frac{d_2}{f} & d_1 + d_2 - \frac{d_1 d_2}{f} \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{pmatrix}$$

So the general transport matrix for arbitrary drift lengths is:

$$\boxed{T = \begin{pmatrix} 1 - \frac{d_2}{f} & d_1 + d_2 - \frac{d_1 d_2}{f} \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{pmatrix}}$$

The focusing element is placed at the center of the gap separating the two accelerating stages, i.e.,

$$d_1 = d_2 = d.$$

Then the matrix simplifies to:

$$T = \begin{pmatrix} 1 - \frac{d}{f} & 2d - \frac{d^2}{f} \\ -\frac{1}{f} & 1 - \frac{d}{f} \end{pmatrix}$$

The transport matrix is a (2x2) matrix, which can be expressed as:

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

To refocus the beam at the downstream stage entrance with the same size, we require both $B = 0$ and $abs(A) = 1$ (that is because $x_f = Ax_i + Bx'_i$, and we want $x_f = x_i$). Note also that $A = -1$ simply means that the image is flipped compared to the object, in the case of a axi-symmetric beam, this has no importance.

$$B = 0 \quad \Rightarrow \quad 2d - \frac{d^2}{f} = 0 \quad \Rightarrow \quad f = \frac{d}{2}.$$

Step 3: Express d in terms of total gap L
Since the total gap length is $L = d_1 + d_2 = 2d$, we have:

$$d = \frac{L}{2} \quad \Rightarrow \quad f = \frac{L}{4}.$$

Step 4: Quadrupole gradient
The focal length of a thin quadrupole is related to its gradient G by:

$$\frac{1}{f} = \frac{GL_q}{B\rho} \quad \Rightarrow \quad G = \frac{B\rho}{fL_q}.$$

Substituting $f = L/4$:

$$\boxed{G = \frac{4\,B\rho}{L\,L_q}}.$$

For ultra-relativistic ($E_{kin} >> E_{rest}$ electrons ($q = 1$) ($B\rho \approx p/q \approx E/c$):

$$\boxed{G = \frac{4\,E}{c\,q\,L\,L_q}}.$$

This expresses the required quadrupole gradient in terms of the electron energy E, total gap length L, and quadrupole length L_q.

(b) Numerical Example

Given:

$$E = 10 \text{ GeV}, \quad L = 1.0 \text{ m}, \quad L_q = 0.10 \text{ m}.$$

Using the following formula give in the instructions, we find that:

$$B\rho \approx 3.335 \times 1 \approx 33.35 \text{ T} \cdot \text{m}.$$

Then

$$G = \frac{4 \cdot 33.35}{1.0 \cdot 0.10} \approx 1333 \text{ T/m}.$$

Comparison: - Typical normal-conducting quadrupoles: ~ 10 T/m.
$\Rightarrow$ The required gradient of ~ 1.33 kT/m cannot be reach by typical quadrupole magnets, we need to find an alternative...

(d) Plasma Lens Alternative

An **active plasma lens (APL)** produces a symmetric magnetic field (no need of a quadrupole triplet!) that focuses the beam equally in both transverse planes. The focusing gradient can be approximated as

$$g_{\mathrm{APL}}\,[\mathrm{T/m}] = 200\,\frac{I\,[\mathrm{kA}]}{(R\,[\mathrm{mm}])^2},$$

where

- I is the discharge current through the plasma column
- R is the effective plasma lens radius.

$$I[\mathrm{kA}] = \frac{(R[\mathrm{mm}])^2}{200}\,\mathrm{G[T/m]}$$

Computation for the given values :
For $R = 0.5\,\mathrm{mm}$ and $G = 1.33\,\mathrm{kT/m}$:

$$I = \frac{0.5^2}{200}\,1333 \approx 1.67\,\mathrm{kA}.$$

Thus, a current of order $\sim 1.67\,\mathrm{kA}$ is needed to provide the required focusing strength.

(e) Scaling with Beam Energy

Since $B\rho \propto E$, the required plasma-lens current scales linearly with beam energy:

$$I \propto B\rho \propto E.$$

At TeV-scale energies, the discharge currents required for the active plasma lenses become much larger if the gap length remains the same. In practice, we would increase the gap length, as well as the length of the plasma lens instead.

To go further, one can try to understand whether the aperture of the plasma lens would need to become larger when increasing the drift length due to larger witness energies. To do so, you can use the matched size equation and the envelope equation. Note that while the normalized emmitance is preserved under acceleration, the geometric emittance decreases.

3. Single stage acceleration schemes

(a) The proton bunch has a length of $\sigma_z \sim 5.1\,\mathrm{cm}$ and a transverse size $\sigma_r \sim 200\,\mu\mathrm{m}$. Calculate the wave-breaking field amplitude of the plasma, knowing that, ideally, $\sigma_z \sim \lambda_{pe}$. Comment on the result.

The plasma frequency is given by

$$\omega_{pe} = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}, \qquad k_{pe} = \frac{\omega_{pe}}{c}, \qquad \lambda_{pe} = \frac{2\pi}{k_{pe}} = \frac{2\pi c}{\omega_{pe}}.$$

The maximum (cold non-relativistic) wave-breaking field is

$$E_{wb} = \frac{m_e c\,\omega_{pe}}{e}.$$

If we assume $\sigma_z \sim \lambda_{pe}$, then

$$\lambda_{pe} = 0.051\,\mathrm{m} \quad \Rightarrow \quad \omega_{pe} = \frac{2\pi c}{\lambda_{pe}} = \frac{2\pi(3.0\times10^8)}{0.051} = 3.7\times10^{10}\,\mathrm{s}^{-1}.$$

Hence, the plasma density is

$$n_e = \frac{\varepsilon_0 m_e \omega_{pe}^2}{e^2} = 4.3\times10^{17}\,\mathrm{m}^{-3} = 4.3\times10^{11}\,\mathrm{cm}^{-3}.$$

The wave-breaking field is then

$$E_{wb} = \frac{m_e c \omega_{pe}}{e} = 6.3\times10^7\,\mathrm{V/m} = 63\,\mathrm{MV/m}.$$

Comment: The estimated $E_{wb} \simeq 63\,\mathrm{MV/m}$ is not so interesting compared to conventional accelerator gradients ($10 - 100\,\mathrm{MV/m}$ typically). Using the proton bunch from SPS as is is not so interesting for driving wakefields, this is why we rely on the self-modulation process (see Marlene's lecture for more details). What is important to note, is that if we use the self-modulation process, then the condition to satisfy is given in (b), i.e., $\frac{1}{k_{pe}} > \sigma_r$.

(b) Calculate now the wave-breaking field amplitude if the condition to fulfill is $1/k_{pe} > \sigma_r$ instead. Comment on the result.

Using $1/k_{pe} = \sigma_r = 2.0\times10^{-4}\,\mathrm{m}$, we find

$$k_{pe} = \frac{1}{\sigma_r} = 5.0\times10^3\,\mathrm{m}^{-1}, \qquad \omega_{pe} = k_{pe}c = \frac{c}{\sigma_r} = \frac{3.0\times10^8}{2.0\times10^{-4}} = 1.5\times10^{12}\,\mathrm{s}^{-1}.$$

The corresponding plasma density is

$$n_e = \frac{\varepsilon_0 m_e \omega_{pe}^2}{e^2} = 7.1\times10^{20}\,\mathrm{m}^{-3} = 7.1\times10^{14}\,\mathrm{cm}^{-3}.$$

The wave-breaking field becomes

$$E_{wb} = \frac{m_e c \omega_{pe}}{e} = 2.6\times10^9\,\mathrm{V/m} = 2.6\,\mathrm{GV/m}.$$

Comment: When fulfilling this condition, the plasma can now sustain waves of amplitudes much more interesting for a compact accelerator ($\sim$ 100 times larger than RF cavities typically).