Introduction to gravitational waves

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Conventions

In these notes we use the generalized summation convention: repeated indices are summed over, unless explicitly mentioned otherwise.

We use greek indices $\kappa, \lambda, \mu, ... = (0, 1, 2, 3)$ to denote components of 4-dimensional space-time vectors, and latin indices $i, j, k, ... = (1, 2, 3)$ to denote components 3-dimensional spatial vectors.

Discussing physics in a flat background space-time, we generally use the Minkowski metric and its inverse $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ to raise and lower indices on vectors and tensors.

Partial derivatives are frequently denoted by the short-hand notation $\partial_\mu = \partial/\partial x^\mu$, but for 3-dimensional spatial gradients and divergences we use the symbol ∇ with $\nabla_i = \partial_i$. The 4-dimensional d'Alembert or wave operator is $\Box = \partial^\mu \partial_\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$ and the 3-dimensional laplacean is $\Delta = \nabla^2 = \nabla_i \nabla_i$.

For ease of notation most of the time we employ units in which the velocity of light $c = 1$. Occasionally we reinstate explicit powers of c to facilitate the evaluation of dimensionful observable quantities.

1 Linearized General Relativity

General Relativity, Einstein's theory of gravity, can be derived and motivated along two complementary tracks. It can be considered to be a theory of the dynamical geometry of space-time using concepts like metrics, connections and curvature. Alternatively it can be derived as the field theory of self-interacting spin-2 fields in a fixed Minskowski background. This field theory turns out to be highly non-linear, requiring an infinite series of interaction terms which in the end, under fairly general assumptions, uniquely reproduce the geometric theory. The geometric formulation therefore provides by far the most concise and convenient framework for producing general statements about gravity and its implications for the universe at large, especially in large-curvature environments.

In contrast small-curvature fluctuations propagating on a Minkowski background provide the setting for the description of gravitational waves as measured by present terrestrial and space-borne detectors. Even though such waves may be emitted by strongly interacting systems such as coalescing compact binaries (e.g., black holes, neutron stars or white dwarfs), they are observed in an asymptotic flat environment where they behave like linear spin-2 quadrupole waves. These waves propagate at the speed of light and accordingly they have only two transverse polarization modes with helicities ± 2.

The Lorentz-covariant field equation of a symmetric massless spin-2 field $h_{\mu\nu}$ in Minkowski space-time with metric $\eta_{\mu\nu}$ reads

$$\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h^\lambda{}_\lambda - \eta_{\mu\nu} \left(\Box h^\lambda{}_\lambda - \partial^\kappa \partial^\lambda h_{\kappa\lambda} \right) = -\kappa T_{\mu\nu}. \tag{1}$$

Here $T_{\mu\nu}$ is the divergence-free energy-momentum tensor of matter and radiation which act as sources for the gravitational field, and κ is the coupling constant, related to Newton's constant of gravity and the velocity of light by

$$\kappa^2 = \frac{8\pi G}{c^4} \simeq 2.1 \times 10^{-43}\,\mathrm{kg}^{-1}\ \mathrm{m}^{-1}\ \mathrm{s}^2. \tag{2}$$

In the following we will without further notice use units in which $c = 1$. In the geometrical framework the field h represents a fluctuation of the metric in a Minkoswki background of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}, \tag{3}$$

and the left-hand side of eq. (1) represents the linearized Einstein curvature tensor.

Equation (1) is invariant under abelian gauge transformations

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu, \tag{4}$$

parametrized by the four-vector field ξ_μ. This is a linearized form of general co-ordinate transformations and a necessary counterpart of energy-momentum conservation as the condition

$$\partial^\mu T_{\mu\nu} = 0 \tag{5}$$

requires the divergence of the left-hand side of eq. (1) to vanish.

The field equation (1) can be simplified by switching to a different set of field variables defined by

$$\underline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} h^{\lambda}_{\ \lambda}. \tag{6}$$

In terms of these field components the equation takes the form

$$\Box\, \underline{h}_{\mu\nu} - \partial_\mu \partial^\lambda \underline{h}_{\lambda\nu} - \partial_\nu \partial^\lambda \underline{h}_{\lambda\mu} + \eta_{\mu\nu} \partial^\kappa \partial^\lambda \underline{h}_{\kappa\lambda} = -\kappa T_{\mu\nu}, \tag{7}$$

which is invariant under modified gauge transformations

$$\underline{h}'_{\mu\nu} = \underline{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial^\lambda \xi_\lambda. \tag{8}$$

Gauge fixing is the choice of a smooth set of representatives from all classes of gauge-equivalent fields $\underline{h}_{\mu\nu}$; a convenient choice is obtained by imposing the De Donder gauge

$$\partial^\mu \underline{h}_{\mu\nu} = 0, \tag{9}$$

which reduces the field equation further to the linear inhomogeneous wave equation

$$\Box\, \underline{h}_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{10}$$

That it is possible to impose the condition (9) is seen by observing that for any solution $\underline{h}_{\mu\nu}$ of the field equation a gauge transformation can cancel its divergence:

$$\partial^\mu \underline{h}'_{\mu\nu} = \partial^\mu \underline{h}_{\mu\nu} + \Box\, \xi_\nu = 0, \tag{11}$$

provided one takes the gauge parameters to be a solution of the equation

$$\Box\, \xi_\nu = -\partial^\mu \underline{h}_{\mu\nu}. \tag{12}$$

Observe that the gauge condition (9) makes the wave equation (10) compatible with the condition (5) for energy-momentum conservation.

In regions where the energy-momentum tensor of matter vanishes: $T_{\mu\nu} = 0$, one can further eliminate the trace of the gravitational field:

$$\underline{h}^{\lambda}_{\ \lambda} = -h^{\lambda}_{\ \lambda} = 0, \tag{13}$$

by performing a residual gauge transformation

$$\underline{h}'^{\lambda}_{\ \lambda} = \underline{h}^{\lambda}_{\ \lambda} - 2\partial^\lambda \xi'_\lambda = 0, \tag{14}$$

where ξ'_λ is to satisfy the conditions

$$\partial^\lambda \xi'_\lambda = \frac{1}{2}\, \underline{h}^{\lambda}_{\ \lambda}, \qquad \Box\, \xi'_\lambda = 0. \tag{15}$$

The first condition cancels the trace of $\underline{h}^{\lambda}_{\ \lambda}$, the second conditon implies by eq. (11) that the field-divergence remains zero: $\partial^\mu \underline{h}'_{\mu\nu} = 0$. Obviously these conditions are compatible only in regions where the trace satisfies a source-free wave equation:

$$\Box\, \underline{h}^{\lambda}_{\ \lambda} = 0 \quad \Leftrightarrow \quad T^{\lambda}_{\ \lambda} = 0. \tag{16}$$

Therefore in such regions the fields $\underline{h}_{\mu\nu}$ and $h_{\mu\nu}$ can be made to coincide.

2 Free field modes

It is instructive to first consider free propagating waves with Fourier decomposition

$$\underline{h}_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^2}\, \varepsilon_{\mu\nu}(k) e^{-ik\cdot x}. \tag{17}$$

As the gravitational field components are real functions of the space-time co-ordinates x^μ the plane-wave amplitudes $\varepsilon_{\mu\nu}(k)$ must satisfy

$$\varepsilon^*_{\mu\nu}(k) = \varepsilon_{\mu\nu}(-k). \tag{18}$$

The full free-field equation (7) then translates to

$$k^2 \varepsilon_{\mu\nu} - k_\mu k^\lambda \varepsilon_{\lambda\nu} - k_\nu k^\lambda \varepsilon_{\lambda\mu} + \eta_{\mu\nu} k^\kappa k^\lambda \varepsilon_{\kappa\lambda} = 0, \tag{19}$$

which is invariant under gauge transformations in wave-vector space of the form

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu \alpha_\nu + k_\nu \alpha_\mu - \eta_{\mu\nu} k^\lambda \alpha_\lambda. \tag{20}$$

The vectors $\alpha_\mu(k)$ introduced here are wave-vector dependent gauge parameters also satisfying a reality condition

$$\alpha^*_\mu(k) = -\alpha_\mu(-k). \tag{21}$$

These gauge transformations can be used to transform amplitudes off the light-cone to a transverse form in 4-dimensional space-time:

$$k^\mu \varepsilon'_{\mu\nu} = 0, \tag{22}$$

by taking

$$\alpha_\nu = -\frac{k^\mu}{k^2}\, \varepsilon_{\mu\nu}, \qquad k^2 \neq 0. \tag{23}$$

It then follows immediately by eq. (19) that off the light-cone all amplitudes vanish, and

$$k^2 \varepsilon'_{\mu\nu} = 0 \quad \Rightarrow \quad \varepsilon'_{\mu\nu}(k) = e_{\mu\nu}(\mathbf{k}, \omega)\, \delta(k^2), \tag{24}$$

with the light-cone defined by

$$k^2 = 0 \quad \Leftrightarrow \quad k^0 = \pm\, \omega_{\mathbf{k}} = \pm\sqrt{\mathbf{k}^2}. \tag{25}$$

The reality condition (18) then becomes

$$e^*_{\mu\nu}(\mathbf{k}, \omega_{\mathbf{k}}) = e_{\mu\nu}(-\mathbf{k}, -\omega_{\mathbf{k}}). \tag{26}$$

Inserting the light-cone restriction (24) into the plane-wave decomposition (17) we get

$$\begin{aligned} \underline{h}_{\mu\nu}(x) &= \int \frac{d^3k}{8\pi^2\omega_{\mathbf{k}}} \left[e_{\mu\nu}(\mathbf{k}, \omega_{\mathbf{k}}) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} + e_{\mu\nu}(\mathbf{k}, -\omega_{\mathbf{k}}) e^{-i(\mathbf{k}\cdot\mathbf{x} + \omega_{\mathbf{k}} t)} \right] \\ &= \int \frac{d^3k}{8\pi^2\omega_{\mathbf{k}}} \left[e_{\mu\nu}(\mathbf{k}, \omega_{\mathbf{k}}) e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} + e^*_{\mu\nu}(\mathbf{k}, \omega_{\mathbf{k}}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}} t)} \right]. \end{aligned} \tag{27}$$

As the light-cone fields are still subject to the transversality condition (22) we require

$$k^{\mu} e_{\mu 0} = \omega_{\mathbf{k}} e_{00} + k_i e_{i0} = 0, \qquad k^{\mu} e_{\mu i} = \omega_{\mathbf{k}} e_{0i} + k_j e_{ji} = 0. \tag{28}$$

Observe that these conditions are still invariant under gauge transformations (20) restricted to the light-cone as well:

$$\alpha_{\mu}(k) = a_{\mu}(\mathbf{k}, \omega_{\mathbf{k}})\, \delta(k^2), \tag{29}$$

such that

$$\begin{aligned} e'_{00} &= e_{00} - \omega_{\mathbf{k}} a_0 + \mathbf{k} \cdot \mathbf{a}, \quad e'_{0i} = e_{0i} - \omega_{\mathbf{k}} a_i + k_i a_0, \\ e'_{ij} &= e_{ij} + k_i a_j + k_j a_i - \delta_{ij} \left(\omega_{\mathbf{k}} a_0 + \mathbf{k} \cdot \mathbf{a} \right). \end{aligned} \tag{30}$$

These transformations can be used to impose 3-dimensional transversality and tracelessness of the light-cone amplitudes, as follows. Observe that by the first condition (28) and the light-cone property $\omega_{\mathbf{k}}^2 = \mathbf{k}^2$ the combination $e_{i0} + k_i e_{00}/\omega_{\mathbf{k}}$ is transverse:

$$k_i \left(e_{i0} + \frac{k_i}{\omega_{\mathbf{k}}} e_{00} \right) = 0. \tag{31}$$

As therefore expected, under a gauge transformation its change is proportional to the transverse components of $\mathbf{a}$:

$$e'_{i0} + \frac{k_i}{\omega_{\mathbf{k}}}\, e'_{00} = e_{i0} + \frac{k_i}{\omega_{\mathbf{k}}} e_{00} - \omega_{\mathbf{k}} \left(a_i - \frac{k_i k_j}{\omega_{\mathbf{k}}^2}\, a_j \right). \tag{32}$$

Next there are two scalar combinations transforming as

$$\begin{aligned} e'_{00} + e'_{ii} &= e_{00} + e_{ii} - 4\omega_{\mathbf{k}} a_0, \\ e'_{00} - \frac{1}{3}\, e'_{ii} &= e_{00} - \frac{1}{3}\, e_{ii} + \frac{4}{3}\, \mathbf{k} \cdot \mathbf{a}. \end{aligned} \tag{33}$$

Thus it follows that by judicious choice of a_0 and $\mathbf{a}$ we can eliminate e'_{00}, e'_{i0} and e'_{ii}. Take

$$\begin{aligned} & a_0 = \frac{1}{4\omega_{\mathbf{k}}} \left(e_{00} + e_{ii} \right), \qquad \frac{\mathbf{k} \cdot \mathbf{a}}{\omega_{\mathbf{k}}} = \frac{1}{4\omega_{\mathbf{k}}} \left(e_{ii} - 3 e_{00} \right), \\ & a_i - \frac{k_i k_j}{\omega_{\mathbf{k}}^2}\, a_j = \frac{1}{\omega_{\mathbf{k}}} \left(e_{i0} + \frac{k_i}{\omega_{\mathbf{k}}}\, e_{00} \right); \end{aligned} \tag{34}$$

then we get as a result

$$e'_{00} = 0, \quad e'_{i0} = 0, \quad e'_{ii} = 0, \tag{35}$$

which leaves the second constraint (28) in the form

$$k_j e'_{ji} = 0. \tag{36}$$

For example, if the wave vector is directed in the z-direction: $\mathbf{k} = (0, 0, \omega)$, then the amplitudes take the form

$$e'_{\mu\nu}(\mathbf{k}, \omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_+(\omega) & e_\times(\omega) & 0 \\ 0 & e_\times(\omega) & -e_+(\omega) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{37}$$

With this choice of amplitudes the free field satisfies

$$\Box \underline{h}_{\mu\nu} = 0, \quad \partial^\mu \underline{h}_{\mu\nu} = 0, \quad \underline{h}^\lambda{}_\lambda = 0 \tag{38}$$

and therefore $h_{\mu\nu} = \underline{h}_{\mu\nu}$, subject to the additional constraints

$$\underline{h}_{00} = \underline{h}_{i0} = \underline{h}_{ii} = 0, \quad \nabla_j \underline{h}_{ji} = 0. \tag{39}$$

As is manifest from the specific representation (37) such a field has only two independent physical degrees of freedom. In the literature the conditions (39) are referred to as the transverse traceless or TT-gauge, and the corresponding field components are often denoted by $h^{TT}_{\mu\nu}$.

3 Emission of quadrupole waves

We now turn to solving the wave equation (10) in the far field regime, meaning at large distance from the sources, in vacuum and in a Minkowski background space-time. A large distance here is a distance at which only components falling off no faster than $1/r$ survive.

For a start we can write down a formal exact solution of the wave equation using the standard retarded Green's function:

$$\underline{h}_{\mu\nu}(\mathbf{x}, t) = \frac{\kappa}{4\pi} \int_{S_r} d^3x' \, \frac{T_{\mu\nu}(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|}. \tag{40}$$

As the sources where $T_{\mu\nu} \neq 0$ are supposedly localized at a large distance from the point labeled $\mathbf{x}$, where the field is evaluated, the integral is taken over a large sphere S_r containing the sources, with the origin fixed in some well-defined internal point of the source region. This sphere thus encloses all of the sources, its radius $r = |\mathbf{x}|$ being much larger than any typical dimension of the source. For example, for a binary star system of maximal extension d the origin may be taken at a fixed point inside the orbit while requiring at all times $r \gg d$.

With these assumptions we can expand the integrand in powers of $1/r$ and neglect all terms of order $1/r^p$ with $p > 1$. This results in the simpler integral

$$\underline{h}_{\mu\nu}(\mathbf{x}, t) = \frac{\kappa}{4\pi r} \int_{S_r} d^3x' \, T_{\mu\nu}(\mathbf{x}', t - r). \tag{41}$$

From this expression it is straighforward to show that the time components of the field $\underline{h}_{0\mu}$ are non-dynamical, representing static fields that do not contribute to the flux of energy and momentum across the surface of the sphere S_r. More precisely, using energy-momentum conservation for the source the components $\underline{h}_{0\mu}$ do not depend on co-ordinate time $t = x^0$:

$$\begin{aligned} \partial_0 \underline{h}_{0\mu} &= \frac{\kappa}{4\pi r} \int_{S_r} d^3x'\, \partial_0 T_{0\mu} = \frac{\kappa}{4\pi r} \int d^3x'\, \nabla_i' T_{i\mu} \\ &= \frac{\kappa}{4\pi r} \oint_{\partial S_r} d^2\sigma\, \hat{r}_i T_{i\mu} = 0. \end{aligned} \tag{42}$$

Here we have used Gauss' theorem to turn the volume integral into a surface integral over the spherical surface ∂S_r, and $\hat{r}_i\, T_{i\mu}$ is a normal component of the energy-momentum tensor, $\hat{\mathbf{r}}$ being the normal unit vector pointing out of the sphere. In view of the assumption that $T_{\mu\nu} = 0$, except for a finite region near the center of the sphere, the vanishing of the surface integral is obvious.

In view of this the only relevant field components for the gravitational energy-momentum flux are the spatial components $\underline{h}_{ij}$. As we are evaluating the wave field in empty space far from the sources we can transform these asymptotic components to the TT-gauge (39) with a gauge transformation that does not affect the source region. It then follows that we can write the large-distance solution (41) in the form

$$\underline{h}_{ij}(\mathbf{x}, t) = \frac{\kappa}{4\pi r}\, (\delta_{ik} - \hat{r}_i \hat{r}_k)\, (\delta_{jl} - \hat{r}_j \hat{r}_l) \left(I_{kl} + \frac{1}{2}\, \delta_{kl}\, \hat{r} \cdot I \cdot \hat{r} \right), \tag{43}$$

where defining the retarded time $u = t - r$

$$I_{ij}(u) = \int_{S_r} d^3x' \left(T_{ij} - \frac{1}{3}\, \delta_{ij} T_{kk} \right) (\mathbf{x}', u)\,, \tag{44}$$

and as before the notation $I_{rr} = \hat{r}_m I_{mn} \hat{r}_n$ denotes its double-radial component. Noting that $I_{ii} = 0$ it is straightforward to check that

$$\hat{r}_j \underline{h}_{ji} = 0, \qquad \underline{h}_{ii} = 0, \tag{45}$$

proving that the expression (43) represents the transverse and traceless components of the wave fields (41).

Using a standard trick the expression for I_{ij} can be rewritten for compact sources in terms of the second moment of the source energy density:

$$I_{ij}(u) = \frac{1}{2}\, \partial_0^2 \int_{S_r} d^3x' \left(x_i' x_j' - \frac{1}{3}\, \delta_{ij}\, \mathbf{x}'^2 \right) T_{00}(\mathbf{x}', u). \tag{46}$$

The proof uses energy-momentum conservation of the source twice:

$$\partial_0^2\, T_{00} = \partial_0 \nabla_i T_{i0} = \nabla_j \nabla_i T_{ij}, \tag{47}$$

and then involves two partial integrations in (46) to reobtain equation (44). Finally for non-relativistic sources the energy density is dominated by the mass-density $\rho(\mathbf{x}, t)$, which allows us to replace the integral in (46) by the mass quadrupole moment:

$$I_{ij} = \frac{1}{2} \frac{\partial^2 Q_{ij}}{\partial t^2}, \qquad Q_{ij}(u) \simeq \int_{S_r} d^3x' \left(x'_i x'_j - \frac{1}{3} \delta_{ij} \mathbf{x}'^2 \right) \rho(\mathbf{x}', u). \tag{48}$$

Thus we get the final expression for the wave field $\underline{h}_{ij}$ for non-relativistic sources in the TT-gauge:

$$\underline{h}_{ij}(\mathbf{x}, t) = \frac{\kappa}{8\pi r} \left(\delta_{ik} - \hat{r}_i \hat{r}_k \right) \left(\delta_{jl} - \hat{r}_j \hat{r}_l \right) \frac{\partial^2}{\partial t^2} \left(Q_{kl} + \frac{1}{2} \delta_{kl} \, \hat{r} \cdot Q \cdot \hat{r} \right)_{u=t-r}. \tag{49}$$

For the dynamical metric fluctuations $\delta g_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, recalling eqs. (3) and (2), this implies

$$\begin{aligned} &\delta g_{00} = \delta g_{0i} = 0; \\ &\delta g_{ij} = \frac{2G}{r} \left(\delta_{ik} - \hat{r}_i \hat{r}_k \right) \left(\delta_{jl} - \hat{r}_j \hat{r}_l \right) \frac{\partial^2}{\partial t^2} \left(Q_{kl} + \frac{1}{2} \delta_{kl} \, \hat{r} \cdot Q \cdot \hat{r} \right)_{u=t-r}. \end{aligned} \tag{50}$$

4 Flux of energy and angular momentum

The wave equation (10) can be derived as an extremum of the action

$$S = \int d^4x \, \mathcal{L}[h] = \int d^4x \left[-\frac{1}{2} \left(\partial_\lambda \underline{h}_{\mu\nu} \right)^2 + \kappa \, \underline{h}^{\mu\nu} T_{\mu\nu} \right], \tag{51}$$

such that

$$\frac{\delta S}{\delta \underline{h}^{\mu\nu}} = 0 \quad \Rightarrow \quad \Box \, \underline{h}_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{52}$$

Requiring energy-momentum conservation of matter, the wave equation implies

$$-\kappa \, \partial^\mu T_{\mu\nu} = \Box \, \partial^\mu \underline{h}_{\mu\nu} = 0, \tag{53}$$

which shows that by imposing the De Donder gauge condition (9) we consistently select a special set of solutions of the wave equation. The action can be converted to the hamiltonian form by defining

$$\pi_{\mu\nu} = \frac{\delta S}{\delta \, \partial_t \underline{h}^{\mu\nu}} = \partial_t \underline{h}_{\mu\nu}, \tag{54}$$

and performing a Legendre transformation

$$\mathcal{H} = \partial_t \underline{h}^{\mu\nu} \pi_{\mu\nu} - \mathcal{L}[h] = \frac{1}{2} \pi_{\mu\nu}^2 + \frac{1}{2} \left(\nabla \underline{h}_{\mu\nu} \right)^2 - \kappa \underline{h}^{\mu\nu} T_{\mu\nu}. \tag{55}$$

The field equations in hamiltonian form then read

$$\partial_t \underline{h}_{\mu\nu} = \frac{\partial \mathcal{H}}{\partial \pi^{\mu\nu}} = \pi_{\mu\nu}, \qquad \partial_t \pi_{\mu\nu} = -\frac{\partial \mathcal{H}}{\partial \underline{h}^{\mu\nu}} = \Delta \underline{h}_{\mu\nu} + \kappa T_{\mu\nu}. \tag{56}$$

It also follows that the energy density of the gravitational field is expressed by

$$\mathcal{E}[h] = \frac{1}{2} \left[\left(\partial_t \underline{h}_{\mu\nu} \right)^2 + \left(\nabla \underline{h}_{\mu\nu} \right)^2 \right] - \kappa \underline{h}^{\mu\nu} T_{\mu\nu}, \tag{57}$$

where the term with the energy-momentum tensor describes the interaction energy of gravity with matter. There is an associated field momentum $\mathbf{\Pi}$ to be identified with the energy flux of gravitational waves:

$$\mathbf{\Pi} = -\nabla \underline{h}^{\mu\nu} \partial_t \underline{h}_{\mu\nu}, \tag{58}$$

such that an equation of continuity holds in the form

$$\partial_t \mathcal{E} = -\nabla \cdot \mathbf{\Pi} - \kappa \, \underline{h}^{\mu\nu} \partial_t T_{\mu\nu}. \tag{59}$$

For free gravitational waves or gravitational fields with stationary sources it follows that the energy E_V inside a volume V changes only by a flux of gravitational energy across the boundary ∂V:

$$\frac{dE_V}{dt} = \int_V d^3x \, \partial_t \mathcal{E} = -\int_V d^3x \, \nabla \cdot \mathbf{\Pi} = -\oint_{\partial V} d^2\sigma \, \Pi_n, \tag{60}$$

where

$$\Pi_n = \hat{\mathbf{n}} \cdot \mathbf{\Pi}, \tag{61}$$

is the normal component of the field momentum $\mathbf{\Pi}$ at the surface element $d^2\sigma$ of the boundary obtained by the inproduct with the normal unit vector $\hat{\mathbf{n}}$.

In addition to energy and momentum gravitational waves can also transport angular momentum. Indeed, with a given spatial volume V we can associate the 3-dimensional axial vector quantity

$$L_{Vi} = \varepsilon_{ijk} \int_V d^3x \left(2\underline{h}_{jl} \, \partial_t \underline{h}_{kl} - x_j \nabla_k \underline{h}_{mn} \, \partial_t \underline{h}_{mn} \right). \tag{62}$$

In the absence of sources in the volume V the rate of change of L_i is given by:

$$\begin{aligned} \frac{dL_{Vi}}{dt} &= \varepsilon_{ijk} \int_V d^3x \left[2\underline{h}_{jl} \partial_t^2 \underline{h}_{kl} - x_j \nabla_k \underline{h}_{mn} \partial_t^2 \underline{h}_{mn} - \frac{1}{2} x_j \nabla_k \left(\partial_t \underline{h}_{mn} \right)^2 \right] \\ &= \varepsilon_{ijk} \int_V d^3x \left[2\underline{h}_{jl} \Delta \underline{h}_{kl} - x_j \nabla_k \underline{h}_{mn} \Delta \underline{h}_{mn} - \frac{1}{2} x_j \nabla_k \left(\partial_t \underline{h}_{mn} \right)^2 \right] \\ &= \varepsilon_{ijk} \oint_{\partial V} d^2\sigma \, \hat{n}_m \left[\underline{h}_{jl} \overleftrightarrow{\nabla}_m \underline{h}_{kl} - x_j \nabla_k \underline{h}_{nl} \nabla_m \underline{h}_{nl} \right] - \varepsilon_{ijk} \oint_{\partial V} d^2\sigma \, \hat{n}_k x_j \mathcal{L}[h]. \end{aligned} \tag{63}$$

As before $d^2\sigma$ is a surface element on the boundary and $\hat{\mathbf{n}}$ the local normal unit vector. Clearly if the field $\underline{h}_{ij}$ is localized in a finite volume and we take V large enough all boundary terms vanish and we have a conservation law

$$\frac{dL_{Vi}}{dt} = 0. \tag{64}$$

If the fields do not vanish at the boundary an amount of angular momentum is transported across the boundary as given by the surface integral. If there is matter inside the volume V the expression (62) no longer represents the total angular momentum inside the volume V; the angular momentum of the matter will have to be included, and there may be transfer of angular momentum between matter and gravitational radiation. As long as no matter is flowing across the boundary of the integration volume, eq. (63) still represents the contribution to the angular momentum balance inside V arising from transport by gravitational waves across the boundary.

5 Plane waves

The simplest application of the expressions for energy and angular momentum density is provided by free plane waves (27). For ease of evaluation in this section we keep explicit powers of the velocity of light c.

Consider a plane wave in the TT-gauge with wave vector $\mathbf{k} = (0, 0, \omega/c)$ in the z-direction and arbitrary polarization. Decomposing the amplitude it has only components in the x-y-plane of the form

$$h_{ab} = e_{ab} \cos\omega(t - z/c), \qquad e_{ab} = \left(\begin{array}{cc} e_+ & e_\times \\ e_\times & -e_+ \end{array} \right), \qquad a, b = (1, 2). \tag{65}$$

As the energy flow is in the direction of the field momentum (the z-direction), the energy flow per unit area A in the x-y-plane at the point $z = 0$ is given by

$$\left. \frac{dE}{dAdt} \right|_{z=0} = \left. \Pi_z \right|_{z=0} = \left. \partial_t h_{ab} \nabla_z h_{ab} \right|_{z=0} = -\frac{2\omega^2}{c} \left(e_+^2 + e_\times^2 \right) \sin^2 \omega t. \tag{66}$$

Here the minus sign indicates that the energy is lost from the region $z < 0$. The time averaged energy loss over a period $T = 2\pi/\omega$ then is

$$\overline{\frac{dE}{dAdt}} = \frac{1}{T} \int_0^T dt \left. \frac{dE}{dAdt} \right|_{z=0} = -\frac{\omega^2}{c} \left(e_+^2 + e_\times^2 \right). \tag{67}$$

We wish to express this result in terms of deformations of the metric from flat minkowskian geometry; according to eq. (3) this is accomplished by writing

$$a_{ij} = g_{ij} - \delta_{ij} = 2\kappa h_{ij}. \tag{68}$$

In terms of these dimensionless deformations the energy flux reads

$$\overline{\frac{dE}{dAdt}} = -\frac{\omega^2}{4\kappa^2 c} \left(a_+^2 + a_\times^2 \right) = \frac{\pi c^3 f^2}{8G} \left(a_+^2 + a_\times^2 \right), \tag{69}$$

where we have also replaced the angular frequency with the period frequency: $\omega = 2\pi f$. It is easy to check that this quantity has the dimensions of $\mathrm{W/m^2}$. In figure 1 a dimensionless amplitude $h = \sqrt{a_+^2 + a_\times^2}$ has been plotted as a function of frequency f for various values if the power per unit area, ranging from 1 $\mu\mathrm{W/m^2}$ to 1 $\mathrm{MW/m^2}$. The frequency range in which the LIGO (and Virgo) detectors operate is indicated.

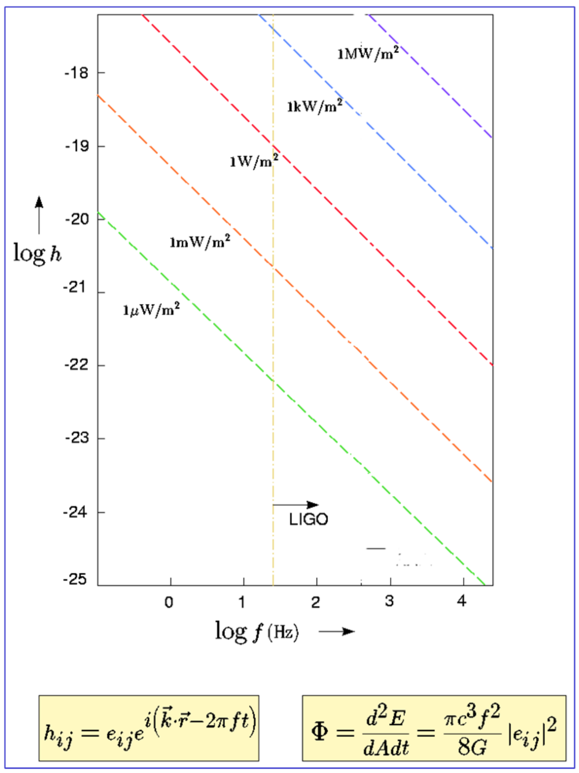

Fig. 1: Deformation amplitudes of plane gravitational waves as a function of frequency f for intensities ranging from 1 $\mu\mathrm{W/m^2}$ - 1 $\mathrm{MW/m^2}$.

The weakness of gravity, or equivalently the stiffness of space, is manifest. For example a wave with frequency $f = 100$ Hz and a large energy flux of 1 $\mathrm{W/m^2}$ deforms space by as little as 2.5×10^{-20}.

Plane waves can also carry angular momentum, but this quantity is associated with circularly polarized waves. To see this, consider again a plane wave moving in the z-direction, but with a fixed phase difference between the linear polarization modes:

$$\underline{h}_{11} = -\underline{h}_{22} = e_+ \cos\omega\left(t - z/c\right), \qquad \underline{h}_{12} = \underline{h}_{21} = e_\times \cos[\omega\left(t - z/c\right) + \alpha]. \tag{70}$$

We compute the transport of the z-component of angular momentum per unit area across the x-y-plane at fixed $z = 0$. The only contribution to this angular momentum flux is

$$\left.\frac{dL_z}{dAdt}\right|_{z=0} = \varepsilon_{ab}\, \underline{h}_{ac} \overleftrightarrow{\nabla}_z \, \underline{h}_{bc}\Big|_{z=0} = -4ke_+e_\times \sin\alpha, \tag{71}$$

with $k = \omega/c$. Again the sign signifies the loss of angular momentum in the region $z < 0$. Clearly the angular momentum transport is maximal for a phase difference $\alpha = \pm\pi/2$ between the two linear polarization modes, corresponding to purely left- or right-rotating amplitudes. The sign of the angular momentum transported in the positive z-direction equals the sign of the phase difference $\alpha \in [-\pi, \pi)$.

As for the other components of angular momentum, there is no net flux of $L_{x,y}$ across the plane $z = 0$.

6 Energy and angular-momentum flow created by matter sources

For practical applications it is convenient to consider isolated sources of gravitational waves, enclose them in a large sphere of radius r and compute the flux of energy in the form of gravitational waves across the spherical surface. Then the surface element in eq. (60) is

$$d^2\sigma = r^2 \sin\theta d\theta d\varphi \equiv r^2 d^2\Omega, \tag{72}$$

with $d^2\Omega$ being the 2-dimensional spatial angle measured as the surface element cut out of the unit sphere by the cone with opening angles (θ, φ). With some abuse of notation we can then define the differential flux of energy in the direction of the cone as

$$\frac{dE}{d^2\Omega dt} = r^2\Pi_r = r^2\partial_t\underline{h}_{\mu\nu}\partial_r\underline{h}_{\mu\nu}. \tag{73}$$

Now if the radial distance r is lage enough, the only components of the gravitational field contributing to the flux are those falling off no faster than $1/r$, which are precisely the retarded quadrupole fields (49). In the TT-gauge these fields depend on time only through the derivatives of the mass quadrupole components and therefore

$$\begin{aligned}
\partial_t\underline{h}_{ij} &= -\frac{\kappa}{8\pi r}\,(\delta_{ik} - \hat{r}_i\hat{r}_k)\,(\delta_{jl} - \hat{r}_j\hat{r}_l)\,\frac{\partial^3}{\partial t^3}\left(Q_{kl} + \frac{1}{2}\,\delta_{kl}\,\hat{r}\cdot Q\cdot\hat{r}\right)_{u=t-r}, \\
\partial_r\underline{h}_{ij} &= \frac{\kappa}{8\pi r}\,(\delta_{ik} - \hat{r}_i\hat{r}_k)\,(\delta_{jl} - \hat{r}_j\hat{r}_l)\,\frac{\partial^3}{\partial t^3}\left(Q_{kl} + \frac{1}{2}\,\delta_{kl}\,\hat{r}\cdot Q\cdot\hat{r}\right)_{u=t-r} + \mathcal{O}(1/r^2),
\end{aligned} \tag{74}$$

with no components $\underline{h}_{0\mu}$ contributing. The differential energy flux at large r is therefore given by

$$\frac{dE}{d^2\Omega dt} = -r^2\left(\partial_t\underline{h}_{ij}\right)^2 = -\frac{\kappa^2}{64\pi^2}\left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{r}\cdot\dddot{Q}^{\,2}\cdot\hat{r} + \frac{1}{2}(\hat{r}\cdot\dddot{Q}\cdot\hat{r})^2\right]. \tag{75}$$

The minus sign here denotes that energy is flowing out of the surface. Reinstating powers of c and recalling the relation (2) of the coupling κ with Newton's constant the result becomes

$$\frac{dE}{d^2\Omega dt} = -\frac{G}{8\pi c^5}\left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{r}\cdot\dddot{Q}^{\,2}\cdot\hat{r} + \frac{1}{2}(\hat{r}\cdot\dddot{Q}\cdot\hat{r})^2\right]. \tag{76}$$

The total energy emitted per unit time by the source is found by integrating the expression (76) over all angles. This calculation can be simplified by the elementary results

$$\frac{1}{4\pi}\int d^2\Omega\,\hat{r}_i\hat{r}_j \equiv \langle\hat{r}_i\hat{r}_j\rangle = \frac{1}{3}\,\delta_{ij}, \qquad \langle\hat{r}_i\hat{r}_j\hat{r}_k\hat{r}_l\rangle = \frac{1}{15}\left(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right). \tag{77}$$

It is then straightforward to show that

$$\frac{dE}{dt} = -\frac{G}{5c^5}\,\mathrm{Tr}\,\dddot{Q}^{\,2}. \tag{78}$$

As for the energy flux we can also write down an expression for the directional angular momentum flux per unit of spherical angle. Observing that $\hat{n}_k = x_k/r$ and $\varepsilon_{ijk}\,x_jx_k/r = 0$, the last term in eq. (63) vanishes and

$$\begin{aligned}\frac{dL_i}{d^2\Omega dt} &= \varepsilon_{ijk}r^2\hat{r}_l\left(2\underline{h}_{jn}\nabla_l\underline{h}_{kn} - x_j\nabla_k\underline{h}_{mn}\nabla_l\underline{h}_{nm}\right)\\ &= \varepsilon_{ijk}r^2\left(2\underline{h}_{jn}\partial_r\underline{h}_{kn} - x_j\nabla_k\underline{h}_{mn}\partial_r\underline{h}_{nm}\right).\end{aligned} \tag{79}$$

Recalling the results (74) we get

$$\frac{dL_i}{d^2\Omega dt} = -\varepsilon_{ijk}r^2\left(2\underline{h}_{jn}\partial_t\underline{h}_{kn} - x_j\nabla_k\underline{h}_{mn}\partial_t\underline{h}_{nm}\right) + \mathcal{O}(1/r). \tag{80}$$

As we always apply the large-r limit we neglect the terms falling off with powers of r, and insert the expression for the quadrupole fields (49). The first term on the right-hand side of (80) then becomes

$$\begin{aligned}2\varepsilon_{ijk}r^2\underline{h}_{jn}\partial_t\underline{h}_{kn} = \frac{\kappa^2}{32\pi^2}\,\varepsilon_{ijk}&\left[\left(\ddot{Q}\cdot\dddot{Q}\right)_{jk} - \left(\ddot{Q}\cdot\hat{r}\right)_j\left(\dddot{Q}\cdot\hat{r}\right)_k\right.\\ &\left. -\,\hat{r}_j\left(\dddot{Q}\cdot\ddot{Q}\cdot\hat{r} - \ddot{Q}\cdot\dddot{Q}\cdot\hat{r} + \ddot{Q}\cdot\hat{r}\,\hat{r}\cdot\dddot{Q}\cdot\hat{r} - \dddot{Q}\cdot\hat{r}\,\hat{r}\cdot\ddot{Q}\cdot\hat{r}\right)_k\right].\end{aligned} \tag{81}$$

To evaluate the second one we use the results

$$\nabla_i r = \hat{r}_i, \qquad \nabla_i\hat{r}_j = \frac{1}{r}\left(\delta_{ij} - \hat{r}_i\hat{r}_j\right); \tag{82}$$

then the leading terms, surviving in the large r limit, are

$$\varepsilon_{ijk}r^2x_j\nabla_k\underline{h}_{mn}\partial_t\underline{h}_{nm} = -\frac{\kappa^2}{32\pi^2}\,\varepsilon_{ijk}\hat{r}_j\left[\dddot{Q}\cdot\ddot{Q}\cdot\hat{r} - \dddot{Q}\cdot\hat{r}\,\hat{r}\cdot\ddot{Q}\cdot\hat{r} + \frac{1}{2}\,\ddot{Q}\cdot\hat{r}\,\hat{r}\cdot\dddot{Q}\cdot\hat{r}\right]_k. \tag{83}$$

Combining these results we get for the direction-dependent angular momentum flux after reinstating powers of c:

$$\frac{dL_i}{d^2\Omega dt} = -\frac{G}{4\pi c^5}\,\varepsilon_{ijk}\left[\left(\ddot{Q}\cdot\dddot{Q}\right)_{jk} - \left(\ddot{Q}\cdot\hat{r}\right)_j\left(\dddot{Q}\cdot\hat{r}\right)_k \right. \\ \left. + \hat{r}_j\left(\ddot{Q}\cdot\dddot{Q}\cdot\hat{r} - \frac{1}{2}\,\ddot{Q}\cdot\hat{r}\,\hat{r}\cdot\dddot{Q}\cdot\hat{r}\right)_k\right]. \tag{84}$$

Finally integrating over all angles using eqs. (77) the total angular momentum flux is found to be

$$\frac{dL_i}{dt} = -\frac{2G}{5c^5}\,\varepsilon_{ijk}\,[\ddot{Q}\cdot\dddot{Q}]_{jk}. \tag{85}$$

7 Newtonian binaries

In this section we consider a newtonian binary star system in circular orbit. The masses of the stars are $m_{1,2}$, and their separation is

$$\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1. \tag{86}$$

We take the center of mass (CM) as the origin of co-ordinates, hence

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0. \tag{87}$$

Then we can convert the positions to the CM frame as

$$\mathbf{r}_1 = -\frac{m_2}{M}\,\mathbf{R}, \qquad \mathbf{r}_2 = \frac{m_1}{M}\,\mathbf{R}. \tag{88}$$

In this frame the newtonian gravitational force is

$$\mu\ddot{\mathbf{R}} = -\frac{G\mu M}{R^2}\,\hat{\mathbf{R}}, \tag{89}$$

where M and μ are the total and reduced mass, respectively:

$$M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{90}$$

We first consider circular orbits in the absence of gravitational radiation. Taking the plane of the orbit to be the x-y-plane, the orbits can be parametrized by

$$\mathbf{R} = R\left(\cos\omega t, \sin\omega t, 0\right). \tag{91}$$

Inserting this into eq. (89) one gets

$$\omega^2 = \frac{GM}{R^3}. \tag{92}$$

In such an orbit the total energy is

$$E = \frac{1}{2}\,\mu \dot{\mathbf{R}}^2 - \frac{G\mu M}{R} = -\frac{G\mu M}{2R}, \tag{93}$$

and the angular momentum is

$$\mathbf{L} = \mu \mathbf{R} \times \dot{\mathbf{R}} = (0, 0, L_z), \qquad L_z = \mu R^2 \omega = \mu \sqrt{GMR}. \tag{94}$$

The mass quadrupole moment of this binary system is

$$\begin{aligned} Q_{ij} &= m_1 \left(r_{1i} r_{1j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_1^2 \right) + m_2 \left(r_{2i} r_{2j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_2^2 \right) \\ &= \frac{\mu R^2}{2} \begin{pmatrix} \cos 2\omega t + \frac{1}{3} & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t + \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}, \end{aligned} \tag{95}$$

and therefore

$$\begin{aligned} \ddot{Q}_{ij} &= -2\mu R^2 \omega^2 \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \dddot{Q}_{ij} &= -4\mu R^2 \omega^3 \begin{pmatrix} -\sin 2\omega t & \cos 2\omega t & 0 \\ \cos 2\omega t & \sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{96}$$

Substitution into eq. (75) then leads to the expression for the differential energy loss of the system by emission of gravitational waves:

$$\frac{dE}{d^2\Omega dt} = -\frac{4G^4 \mu^2 M^3}{\pi c^5 R^5} \left[\cos^2\theta + \frac{1}{4}\, \sin^2 2\left(\varphi - \omega t\right) \sin^4\theta \right]. \tag{97}$$

After integration over the angles the expression for the total energy loss is

$$\frac{dE}{dt} = -\frac{32 G^4 \mu^2 M^3}{5 c^5 R^5}. \tag{98}$$

Similarly by eq. (85) the total loss of angular momentum is

$$\frac{dL_z}{dt} = -\frac{32 G^3 \mu^2 M^2}{5 c^5 R^3} \sqrt{\frac{GM}{R}}. \tag{99}$$

The approach taken here is to assume that the binary system loses only a small fraction of its energy and angular momentum per orbit; thus the system changes only adiabatically and at any time one can describe its motion by an almost stable keplerian orbit, in this case a circular orbit of slowly changing radius. Indeed, from the two formulae (93) and (94) one can calculate this slow orbital evolution:

$$\frac{dE}{dt} = \frac{G\mu M}{2R^2}\frac{dR}{dt}, \qquad \frac{dL_z}{dt} = \frac{\mu}{2}\sqrt{\frac{GM}{R}}\frac{dR}{dt}. \tag{100}$$

Comparing these expressions with the results (98) and (99) we find that on both accounts the orbital change is consistently given by

$$\frac{dR}{dt} = -\frac{64G^3\mu M^2}{5c^5R^3}. \tag{101}$$

Equivalently the angular frequency ω or orbital period $T = 2\pi/\omega$ changes by

$$\frac{d\omega}{dt} = \frac{96G^{5/3}}{5c^5}\,\mu M^{2/3}\omega^{11/3}, \qquad \frac{dT}{dt} = -\frac{192\pi}{5}\left(\frac{2\pi G\mu^{3/5}M^{2/5}}{c^3T}\right)^{5/3}. \tag{102}$$

The last expression has the advantage that it is dimensionless.

Exercises

1. Linearized General Relativity

1. The Riemann-Christoffel connection is defined in terms of the metric by

$$\Gamma_{\mu\nu}^{\ \ \lambda} = \frac{1}{2}\, g^{\lambda\kappa} \left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right).$$

a. Using a metric of the form $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$ show that

$$\begin{aligned} \Gamma_{\mu\nu}^{\ \ \lambda} &= \kappa\eta^{\lambda\kappa} \left(\partial_\mu h_{\nu\kappa} + \partial_\nu h_{\mu\kappa} - \partial_\kappa h_{\mu\nu} \right) + \mathcal{O}(\kappa^2) \\ &= \kappa \left(\partial_\mu h_\nu^{\ \lambda} + \partial_\nu h_\mu^{\ \lambda} - \partial^\lambda h_{\mu\nu} \right) + \mathcal{O}(\kappa^2), \end{aligned}$$

where the Minkowski metric has been used to raise and lower indices.
b. The Riemann curvature tensor is defined by

$$R_{\mu\nu\kappa}^{\quad\ \lambda} = \partial_\mu \Gamma_{\nu\kappa}^{\ \ \lambda} - \partial_\nu \Gamma_{\mu\kappa}^{\ \ \lambda} - \Gamma_{\mu\kappa}^{\ \ \sigma} \Gamma_{\nu\sigma}^{\ \ \lambda} + \Gamma_{\nu\kappa}^{\ \ \sigma} \Gamma_{\mu\sigma}^{\ \ \lambda}.$$

Derive the result

$$R_{\mu\nu\kappa}^{\quad\ \lambda} = \kappa\, \left(\partial_\mu \partial_\kappa h_\nu^{\ \lambda} - \partial_\nu \partial_\kappa h_\mu^{\ \lambda} + \partial^\lambda \partial_\nu h_{\mu\kappa} - \partial^\lambda \partial_\mu h_{\nu\kappa} \right) + \mathcal{O}(\kappa^2),$$

and compute the Ricci tensor $R_{\mu\nu} = R_{\mu\lambda\nu}^{\quad\ \lambda}$ and the Riemann scalar $R = g^{\mu\nu} R_{\mu\nu}$.
c. By definition the Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R.$$

Show that

$$G_{\mu\nu} = \kappa \left(\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial_\lambda h_\nu^{\ \lambda} - \partial_\nu \partial_\lambda h_\mu^{\ \lambda} - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\lambda \partial_\kappa h^{\lambda\kappa} \right) + \mathcal{O}(\kappa^2).$$

Use this expression to derive eq. (1) from the standard Einstein equations taking account of definition (2).
d. Check the invariance of the linearized form of $G_{\mu\nu}$ under gauge transformations

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \mathcal{O}(\kappa).$$

2. a. Show that the definition of $\underline{h}_{\mu\nu}$ in eq. (6) implies

$$h_{\mu\nu} = \underline{h}_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} \underline{h}^\lambda_{\ \lambda}.$$

b. Check the invariance of eq. (7) under the gauge transformations (8).

2. Free field modes

1. a. Check that the transformations of the form

$$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu \alpha_\nu + k_\nu \alpha_\mu - \eta_{\mu\nu} k^\lambda \alpha_\lambda$$

leave the field equation (19) for the wave modes invariant.
b. Show that we can identify $i\alpha_\mu(k)$ with the plane-wave coefficients of the gauge parameters:

$$\xi_\mu(x) = i \int \frac{d^4k}{(2\pi)^2}\, \alpha_\mu(k) e^{-ik\cdot x},$$

and that the reality condition $\alpha^*_\mu(k) = -\alpha_\mu(-k)$ is equivalent with $\xi^*_\mu(x) = \xi_\mu(x)$.

2. a. From the definition $k^2 = \mathbf{k}^2 - k_0^2$ show that

$$\int d^4k\, \delta(k^2) A(k) = \int \frac{d^3k}{2\omega_{\mathbf{k}}} \left[A(\mathbf{k}, k_0 = \omega_{\mathbf{k}}) + A(\mathbf{k}, k_0 = -\omega_{\mathbf{k}}) \right],$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2}$. Explain the various steps.
b. Use this result to derive the expression (27).

3. Emission of quadrupole waves

1. Show that

$$\phi(\mathbf{x}, t) = -\frac{1}{4\pi} \int d^3x'\, \frac{\rho(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|)}{|\mathbf{x}' - \mathbf{x}|}$$

is a solution of the inhomogeneous wave equation

$$\Box\, \phi(\mathbf{x}, t) = \rho(\mathbf{x}, t).$$

Explain why it is called the *retarded* solution.

2. Check that the gravitational wave solution (43) satisfies the conditions (45) of being transverse and traceless.

4. Flux of energy and momentum

1. Consider a theory of a scalar field $\phi(x)$ with lagrangean action

$$S[\phi] = \int_{x_1}^{x_2} d^4x\, \mathcal{L}[\phi, \partial\phi] = \int_{x_1}^{x_2} d^4x \left[-\frac{1}{2}\, \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2}\, \phi^2 - \phi\rho \right],$$

where $x^\mu_{1,2}$ are the boundaries of integration.
a. Show that the variation of the action vanishes: $\delta S = 0$ under any variations $\delta\phi$ fixed at the boundaries if and only if the field equation holds:

$$\left(\Box - m^2 \right) \phi = \rho. \tag{A}$$

b. Derive the expression for the conjugate momentum of field at point x:

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \partial_t \phi} = \partial_t \phi,$$

and perform a Legendre transformation to obtain the hamiltonian density:

$$\mathcal{H} = \partial_t \phi \pi - \mathcal{L}.$$

Show that the hamiltonian takes the form

$$H \equiv \int_V d^3x \, \mathcal{H} = \int_V d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} \left(\nabla \phi \right)^2 + \frac{m^2}{2} \phi^2 + \phi \rho \right],$$

where V is the spatial volume of integration.
c. By considering general infinitesimal variations $\delta\phi(x)$ and $\delta\pi(x)$ in the hamiltonian prove that the field equation (A) is reproduced by the Hamilton equations

$$\partial_t \phi(x) = \frac{\delta H}{\delta \pi(x)}, \qquad \partial_t \pi(x) = -\frac{\delta H}{\delta \phi(x)},$$

allowing for partial integrations.
d. Prove that it follows that keeping the boundary terms from partial integrations

$$\frac{dH}{dt} = \int_V d^3x \, \nabla \cdot \left(\pi \nabla \phi \right).$$

e. Argue from this that defining the field energy density

$$\mathcal{E} = \frac{1}{2} \left(\partial_t \phi \right)^2 + \frac{1}{2} \left(\nabla \phi \right)^2 + \frac{m^2}{2} \phi^2 + \phi \rho,$$

and a momentum density

$$\mathbf{\Pi} = -\nabla \phi \, \partial_t \phi,$$

they satisfy an equation of continuity

$$\partial_t \mathcal{E} = -\nabla \cdot \Pi + \phi \, \partial_t \, \rho,$$

and therefore that for stationary sources[1] the total energy in a volume V changes by the flow of momentum across the boundary:

$$\frac{dE_V}{dt} = \frac{d}{dt} \int_V d^3x \, \mathcal{E} = - \oint_{\partial V} d^2\sigma \, \Pi_n,$$

with Π_n the normal component of $\mathbf{\Pi}$ on the boundary ∂V, and a positive Π_n signifying that energy is flowing out of V.

[1]Sources for which the net change after integration over the volume V vanishes.

2. Define the angular momentum associated with the fields in the volume V:

$$L_{Vi} \equiv \int_V d^3x \, (\mathbf{r} \times \Pi)_i = -\varepsilon_{ijk} \int_V d^3x \, x_j \nabla_k \phi \, \partial_t \phi.$$

using the field equation (A) show that

$$\frac{dL_{Vi}}{dt} = -\int_V d^3x \left[\nabla_m \left(\varepsilon_{ijk} x_j \nabla_k \phi \nabla_m \phi \right) + \nabla_k \left(\varepsilon_{ijk} x_j \mathcal{L}[\phi] \right) + \phi \, \varepsilon_{ijk} x_j \nabla_k \rho \right].$$

Argue that for spherically symmetric density ρ the last term vanishes, and

$$\frac{dL_{Vi}}{dt} = -\varepsilon_{ijk} \oint_{\partial V} d^2\sigma \left(x_j \nabla_k \phi \, \hat{n}_m \nabla_m \phi \, + \hat{n}_k x_j \mathcal{L}[\phi] \right).$$

5. Plane waves

1. Using eq. (69) compute the average energy flux in a plane gravitational wave of frequency $f = 250$ Hz and with a total amplitude $h = \sqrt{a_+^2 + a_\times^2} = 10^{-21}$.

6. Energy and angular momentum flow created by matter sources

1. a. Using the explicit form

$$\hat{\mathbf{r}} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)),$$

prove the identities (77). Check them by computing traces over index pairs (ij).
b. Use these results to derive the expression (78) from (77).
c. Similarly derive eq. (85) from (84).

7. Newtonian binaries

1. Derive the expressions (93) and (94) for the energy and angular momentum of a binary system in the CM frame.

2. Use eq. (76) to derive the expression (97) for the energy loss of a binary system in circular orbit.

3. Derive equation (101) for the shrinking of the orbit both from the energy loss and from the angular momentum loss.

4. a. Consider a binary neutron star system with masses $m_1 = 1.2 M_\odot$ and $m_2 = 1.5 M_\odot$, where $M_\odot = 2 \times 10^{30}$ kg is the solar mass. Compute the loss of energy and angular momentum when the stars are in a circular orbit of radius $R = 10^6$ km.
b. How much does the radius decrease in a year (3.16×10^7 s)?